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Arbitrage and asset market equilibrium in infinite dimensional economies with short-selling and risk-averse expected utilities *

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Abstract

We consider a model with an infinite number of states of nature, von Neumann - Morgenstern utilities, where agents have different probability beliefs and where short sells are allowed. We show that no-arbitrage conditions, defined for finite dimensional asset markets models, are not sufficient to ensure existence of equilibrium in presence of an infinite number of states of nature. However, if the individually rational utility set $\mathcal{U}$ is compact, we obtain an equilibrium. We give conditions which imply the compactness of $\mathcal{U}$. We give examples of non-existence of equilibrium when these conditions do not hold.

Keywords: asset market equilibrium, individually rational attainable allocations, individually rational utility set, no-arbitrage prices, no-arbitrage condition.

JEL Classification: C62, D50, D81,D84,G1.

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1 Introduction

In finite dimensional markets with short-selling, conditions on agents' utilities insuring the existence of equilibria (or equivalent to the existence of equilibria) are by now well understood. In particular they can be interpreted as no-arbitrage conditions. In an uncertainty setting, where agents have different beliefs and different risk aversions, as originally shown by Hart (1974), the no-arbitrage conditions may be interpreted as compatibility of agent's risk adjusted beliefs. There is a huge literature on sufficient and necessary conditions for the existence of equilibria in the line of Hart's model, where the market is complete. In finite dimension, one can refer, for instance to Page (1987), Werner (1987), Nielsen (1989), Page and Wooders (1995, 1996), Allouch (1999), Allouch et al (2000), Won and Yannelis (2008). When the number of assets is infinite, and the market is complete, the no-arbitrage condition used for finite dimension do not imply existence of equilibrium. The standard assumption is to assume that the individually rational utility set is compact (see e.g. Cheng (1991), Brown and Werner (1995), Dana and Le Van (1996), Dana et al (1997), Dana and Le Van (2000), Le Van and Truong Xuan (2001).)[1]
In this paper, we consider a model with an infinite number of states of nature, a finite number of agents and Von Neumann - Morgenstern utilities with different expectations.

More precisely, we consider a model where the utility of agent i is

$$U^i(x^i) = \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i)$$

where π^i is her belief and x^i is her consumption. The commodity space is $l^p(\pi)$ with $p \in \{1, 2, \dots, +\infty\}$. [2]

[1] Chichilnisky and Heal (1993) give a condition which implies the boundedness of the individually rational feasible set in L^2. Since the feasible set is closed, it is therefore weakly compact in L^2.

[2] We use the model proposed by Hart. Investors are interested only in their wealth in the second period. We suppose that the market is complete with an asset system $r^1, r^2, \dots, r^k, \dots$. For each portfolio $z = (z^1, z^2, \dots, z^k, \dots)$, the wealth or investor if state s occurs is

$$w_s = \sum_k z^k r_s^k.$$

Her expected utility is:

$$V(w^1, w^2, \dots, w^s, \dots) = \sum_s \pi_s u(w_s).$$

Since the market is complete, the choice of portfolio is equivalent to the choice of wealth. As in Hart's pioneer paper, we consider the expected utility function on wealth.

When the number of states is finite, following Werner (1987), one can introduce for any agent i the set of useful vectors W^i which is the set of directions by following which the utility does not decrease, to obtain the set of no-arbitrage prices denoted by S^i. The latter set is the set of vectors p which satisfy $p \cdot w > 0$ for any $w \in W^i \setminus \{0\}$. We say that the no-arbitrage condition holds if $\cap_i S^i \neq \emptyset$. When the utility functions are strictly concave, strictly increasing, this condition ensures the compactness of the individually rational allocations set. We can also define no-arbitrage prices as in Dana and Le Van (2010). Their set of no-arbitrage prices is larger than the Werner's no-arbitrage prices set.

However, when the number of states is infinite, the no-arbitrage condition (*à la* Werner or *à la* Dana-Le Van) only ensures the boundedness of the individually rational utility set. We give examples where this condition is satisfied and no equilibrium exists. The strategy is therefore to give assumptions which imply the compactness of the individually rational utility set and hence existence of equilibrium. Our conditions might be considered as among the weakest since we give also examples of non existence of equilibrium when these conditions do not hold.

The paper is organized as follows. In Section 2, we set up the model and define the equilibrium. In section 3, we introduce no-arbitrage conditions. In Section 4, we prove the compactness of the individually rational allocations set or the individually rational utility set. We give examples of non-existence of equilibrium when these conditions fail. Finally, proofs are put in Section Appendix. We mention that our methods of proofs are inspired by the ones in Le Van and Truong Xuan (2001). However, their model rules out the risk-neutral agents. That is not the case in our model.

2 The model

There are m agents indexed by $i = 1, \ldots, m$. The belief of agent i in state s is $\pi_s^i \geq 0$, and $\sum_{s=1}^{\infty} \pi_s^i = 1$. Let us denote by π the mean probability $\frac{1}{m}\sum_i \pi^i$. We first assume:

A0: π^i is equivalent to π^j for any i, j i.e. there exists a number $h > 0$ such that $h \leq \frac{\pi_s^i}{\pi_s^j} \leq \frac{1}{h}$ for all i, j, s.[3]

Under **A0**, without loss of generality, one can assume that $\pi_s^i > 0$ for any i, any s. In this paper, we always suppose that the condition **A0** is satisfied and $\pi_s^i > 0$ for any i, any s.

The consumption set of agent i is $X^i = l^p(\pi)$ with $p \in \{1, 2, \ldots, +\infty\}$ and

[3]With **A0**, the consumptions set of the agents belong to same topological space $l^p(\pi)$. We observe that when all agents have the same belief as in Cheng (1991), then **A0** is satisfied.

agent i has an endowment $e^i \in l^p(\pi)$. We assume that for each agent i there exists a concave, strictly increasing, differentiable [4] function $u^i : \mathbb{R} \to \mathbb{R}$, such that, for any i, the function

$$U^i(x^i) = \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i)$$

is real-valued for any $x^i \in X^i$.

Agent i has $l^p(\pi)$ as consumption set, e^i as initial endowment and U^i as utility function, with $i = 1, \ldots, m$.

Definition 1 *An equilibrium is a list* $\left((x^{*i})_{i=1,\ldots,m}, p^*\right)$ *such that* $x^{*i} \in X^i$ *for every* i *and* $p^* \in l_+^q(\pi) \setminus \{0\}$ *and*
(a) *For any* i, $U^i(x) > U^i(x^{*i}) \Rightarrow \sum_{s=1}^{\infty} p_s^* x_s > \sum_{s=1}^{\infty} p_s^* e_s^i$
(b) $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m e^i$.

Definition 2 *A quasi-equilibrium is a list* $\left((x^{*i})_{i=1,\ldots,m}, p^*\right)$ *such that* $x^{i*} \in X^i$ *for every* i *and* $p^* \in l_+^q(\pi) \setminus \{0\}$ *and*
(a) *For any* i, $U^i(x) > U^i(x^{*i}) \Rightarrow \sum_{s=1}^{\infty} p_s^* x_s \geq \sum_{s=1}^{\infty} p_s^* e_s^i$
(b) $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m e^i$.

In this paper, since the consumption set is the whole space, and the utility functions U^i are continuous, any quasi-equilibrium is an equilibrium (see e.g. Geistdoerfer-Florenzano (1982), Proposition 3)

Define

$$\begin{aligned} a^i &= \inf_x u^{i\prime}(x) = u^{i\prime}(+\infty) \\ b^i &= \sup_x u^{i\prime}(x) = u^{i\prime}(-\infty). \end{aligned}$$

Let I_1 be the set of indexes i such that $a^i < b^i$, and I_2 be the set of indexes such that $a^i = b^i$. I_1 is the set of risk averse agents, I_2 is the set of risk neutral ones.

We now give some definitions.

Definition 3 *1. The individually rational attainable allocations set* $\mathcal{A}$ *is defined by*

$$\mathcal{A} = \{(x^i) \in (l^p(\pi))^m \mid \sum_{i=1}^m x^i = \sum_{i=1}^m e^i \ \text{ and } \ U^i(x^i) \geq U^i(e^i) \text{ for all } i\}.$$

2. The individually rational utility set $\mathcal{U}$ *is defined by*

$$\mathcal{U} = \{(v^1, v^2, ..., v^m) \in \mathbb{R}^m \mid \exists x \in \mathcal{A} \ \ s.t \ \ U^i(e^i) \leq v^i \leq U^i(x^i) \ \ \text{for all } i\}.$$

[4] For simplicity, we assume differentiability. The results still hold by using super-gradients. But the proofs will become tedious.

3 No-arbitrage condition and existence of equilibrium

We will first introduce a notion of no-arbitrage price based on useful vectors introduced by Werner.

A vector $w \in l^p(\pi)$ is *useful* for agent i if for any $x \in X^i$, we have

$$U^i(x + \lambda w) \geq U^i(x),\ \forall \lambda \geq 0,\ \forall x \in X^i$$

or equivalently

$$\sum_{s=1}^{\infty} \pi_s^i u^i(x_s + \lambda w_s) \geq \sum_{s=1}^{\infty} \pi_s^i u^i(x_s),\ \forall \lambda \geq 0,\ \forall x \in X^i.$$

Let W^i denote the set of useful vectors for agent i. By the concavity of utility function, we can check that W^i is a closed convex cone.

Let S^i be the set of vectors p in $l^q(\pi)$ [5] which satisfy $\sum_{s\geq 1} \pi_s^i p(s)w(s) > 0$ for any $w \in W^i \setminus \{\emptyset\}$. As in finite dimension, S^i is called the set of no-arbitrage prices of agent i. One can show (see Dana and Le Van (2010)) that w is useful for agent i if and only if

$$\forall x \in X^i,\ \sum_{s=1}^{\infty} \pi_s^i u^{i\prime}(x_s) w_s \geq 0.$$

Observe that if u^i is strictly concave then for any $x \in X^i$ and $w \in W^i$, the function $\lambda \mapsto U^i(x + \lambda w)$ is strictly increasing. In this case, if $w \in W^i \setminus \{0\}$, then

$$\forall x \in X^i,\ \sum_{s=1}^{\infty} \pi_s^i u^{i\prime}(x_s) w_s > 0.$$

Let us now strengthen the notion of useful vectors. A vector w is *strongly useful* for agent i if, for any $x^i \in X^i$, we have

$$\begin{aligned} (i) \sum_s \pi_s^i u^i(x_s^i + \lambda w_s) &\geq \sum_s \pi_s^i u^i(x_s^i), \forall \lambda \geq 0, \\ \text{and } (ii) \sum_s \pi_s^i u^i(x_s^i + \lambda w_s) &> \sum_s \pi_s^i u^i(x_s^i) \text{ for some } \lambda > 0 \end{aligned}$$

It is easy to check that a vector w is *strongly useful* for agent i if, and only if

$$\forall x^i \in X^i, \exists \bar{\lambda} > 0, \forall \lambda \geq \bar{\lambda}, \sum_s \pi_s^i u^i(x_s^i + \lambda w_s) > \sum_s \pi_s^i u^i(x_s^i)$$

This condition turns out to be equivalent to

$$\sum_s \pi_s^i u^{i\prime}(x_s^i) w_s > 0, \forall x^i \in X^i$$

[5] $\frac{1}{p} + \frac{1}{q} = 1$

We now introduce a no-arbitrage condition.
(**NA**) There exist $(\bar{x}^i)_{i=1}^m \in \Pi_i X^i$, such that

$$\lambda^i \pi_s^i u^{i\prime}(\bar{x}_s^i) = \lambda^j \pi_s^j u^{j\prime}(\bar{x}_s^j) = \lambda^k a_k \pi_s^k,\ \forall s, \forall i \in I_1, \forall j \in I_1, \forall k \in I_2$$

and

$$\forall i \in I_1, a^i < \inf_s u^{i\prime}(\bar{x}_s^i) < \sup_s u^{i\prime}(\bar{x}_s^i) < b^i.$$

Observe that if we define a price $p \in l_+^\infty(\pi)$ by

$$\forall s, p_s = \lambda^i \pi_s^i u^{i\prime}(\bar{x}_s^i) = \lambda^k a_k \pi_s^k, i \in I_1, k \in I_2$$

then we have $p \cdot w > 0$ for any strongly useful vector w.

In the following proposition we show that (**NA**) implies the boundedness of $\mathcal{U}$, and we give sufficient conditions to obtain this no-arbitrage condition.

Proposition 1 *(i) If (**NA**) holds, then $\mathcal{U}$ is bounded.*

(ii) Assume either $u^{i\prime}(-\infty) = +\infty$ for all $i \in I_1$ or $u^{i\prime}(+\infty) = 0$ for all $i \in I_1$.

*(ii.1) If $I_2 = \emptyset$, then no arbitrage condition (**NA**) holds.*

*(ii.1) When $I_2 \neq \emptyset$, no arbitrage condition (**NA**) holds if, and only if, $\pi^i = \pi^j, \forall i \in I_2, \forall j \in I_2$.*

Proof: See Appendix. ■

In infinite dimension, with a vector space L as commodity space, Brown and Werner [4], Dana, Le Van and Magnien [10] assume the compactness of $\mathcal{U}$ and get existence of equilibrium with prices in L'. In this paper, we will prove that when the commodity space is $l^\infty(\pi)$, we get an equilibrium with prices in $l^1(\pi) \subset (l^\infty(\pi))'$.

Theorem 1 *Assume **A0**. Our model has an equilibrium if we add the assumption that $\mathcal{U}$ is compact. If X^i is $l^p(\pi)$ with $1 \leq p < +\infty$ then the equilibrium price p^* is in $l^q(\pi)$. If $p = +\infty$, then $p^* \in l^1(\pi)$.*

Proof: Since $\mathcal{U}$ is compact and X^i is $l^p(\pi)$ there exists an equilibrium $((x^{*i}), p^*)$ (see Dana and al (1997)) with $x^{*i} \in l^p(\pi)$.

When $1 \leq p < +\infty$, the price p^* belongs to $l^q(\pi)$. When $p = \infty$ we will show that the equilibrium price belongs to $l^1(\pi)$. The equilibrium price can be written as $p^* + \phi$ where $p^* \in l^1(\pi)$ and ϕ is a purely finitely additive function. For any i, the equilibrium allocation x^{*i} solves the problem:

$$\max \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i)$$

$$\text{s.t. } \sum_{s=1}^{\infty} p_s^* x_s^i + \phi(x^i) = \sum_{s=1}^{\infty} p_s^* e_s^i + \phi(e^i)$$

From Theorem V.3.1, page 91, in Arrow-Hurwicz-Uzawa in [2], for any i, there exists $\zeta_i > 0$ s.t.

$$\sum_{s=1}^{\infty} \pi_s^i u^i(x_s^{*i}) - \zeta_i(\sum_{s=1}^{\infty} p_s^* x_s^{*i} + \phi(x^{*i})) \geq \sum_{s=1}^{\infty} \pi_s^i u^i(x_s) - \zeta_i(\sum_{s=1}^{\infty} p_s^* x_s + \phi(x^i)).$$

Suppose that $\phi \neq 0$. Since $\phi \geq 0$, then $\phi(\mathbf{1}) > 0$, with $\mathbf{1} = (1, 1, 1, \ldots)$. Define $x^i(N)$ as:

$$x_s^i(N) = x_s^{*i} \text{ with } s = 1, 2, \ldots, N.$$
$$x_s^i(N) = x_s^{*i} - 1 \text{ with } s \geq N + 1.$$

Observe that $x^i(N) \in l^\infty(\pi)$. We have:

$$\sum_{s=1}^{\infty} \pi_s^i u^i(x_s^{*i}) - \zeta_i(\sum_{s=1}^{\infty} p_s^* x_s^{*i} + \phi(x^{*i})) \geq \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i(N)) - \zeta_i(\sum_{s=1}^{\infty} p_s^* x_s^i(N) + \phi(x^i(N))).$$

$\Rightarrow$

$$\sum_{s \geq N+1} \pi_s^i u^i(x_s^{*i}) - \zeta_i(\sum_{s \geq N+1} p_s^* x_s^{*i} + \phi(x^{*i})) \geq \sum_{s \geq N+1} \pi_s^i u^i(x_s^{*i} - 1) - \zeta_i(\sum_{s \geq N+1} p_s^*(x_s^{*i} - 1) + \phi(x^{*i} - \mathbf{1})).$$

$\Rightarrow$

$$\sum_{s \geq N+1} \pi_s^i u^i(x_s^{i*}) - \sum_{s \geq N+1} \pi_s^i u^i(x_s^{*i} - 1) - \zeta^i \sum_{s \geq N+1} p_s^* \geq \zeta^i(\phi(x^{*i}) - \phi(x^{*i} - \mathbf{1})) = \zeta^i \phi(\mathbf{1}).$$

Let $N \to \infty$, the left-hand-side converges to 0. This implies $\phi(\mathbf{1}) \leq 0$: a contradiction. Hence $\phi = 0$. ■

4 Sufficient conditions to obtain the compactness of $\mathcal{U}$

We first consider the case where $I_2 = \emptyset$.

Proposition 2 *Assume **A0**. If $b^i = +\infty$ for all i, then the allocation set $\mathcal{A}$ is $l^1(\pi)$-compact. The set $\mathcal{U}$ is also compact.*

Proof: It is given in Appendix. We prove that $\mathcal{A}$ which is bounded in $l^1(\pi)$ satisfies Dunford-Pettis criterion. Hence it is $\sigma(l^1(\pi), l^\infty(\pi))$-compact. We prove however that a bounded set in $l^1(\pi)$ is compact for the $l^1(\pi)$-topology if and only if it is $\sigma(l^1(\pi), l^\infty(\pi))$-compact. Since for all i, U^i is upper semi-continuous on the projection of $\mathcal{A}$ on the i^{th}-component, the set $\mathcal{U}$ is compact. ■

Proposition 3 *Assume **A0**. If $a^i = 0$ for all i, then $\mathcal{U}$ is compact.*

Proof: See Appendix. ■

Remark 1 *1. In Propositions 2 and 3, condition **(NA)** is satisfied (see Proposition 1).*
*2. Example 1 in Appendix shows that **(NA)** is not sufficient to ensure existence of equilibrium if the conditions in Propositions 2, 3 do not hold. We exhibit a model with two agents. Agent 1 has $a^1 = 0$, $b^1 < +\infty$. Agent 2 has $a^2 > 0$, $b^2 = +\infty$. The assumptions of Propositions 2 and 3 are clearly not satisfied. We still have **(NA)** but no equilibrium exists in this model.*

We now state our main results.

Theorem 2 *Suppose one of these two conditions holds:*

* $a^i = 0$ *for every* i.

* $b^i = +\infty$ *for every* i.

*Then (**NA**) holds, and there exists an equilibrium.*

Proof: This is a direct consequence of Theorem 1, and Propositions 2 and 3. ■

Theorem 3 *Suppose that $I_2 \neq \emptyset$. Suppose that these two conditions hold:*

* $a^i = 0$ *for every* $i \in I_1$.

* $b^i = +\infty$ *for every* $i \in I_1$.

Then we have

$$\textbf{NA} \Leftrightarrow \exists \text{ Equilibrium.}$$

Proof: See Appendix. ■

We can be surprised that in presence of risk-neutral agents we have to impose $u^{i\prime}(+\infty) = 0$ and $u^{i\prime}(-\infty) = +\infty$ for any agent $i \in I_1$. In Appendix we give two examples with two agents, one of them is risk neutral while the second one is risk averse. In Example 2, the risk averse agent has the marginal utility at $-\infty$ equal to $+\infty$. In Example 3, the marginal utility at $+\infty$ of the risk averse agent is 0. In both examples, there exists no equilibrium.

Remark 2 *When there are short-sales constraints, the allocation set $\mathcal{A}$ is compact in $l^p(\pi)$ for any p. Indeed, suppose we have short-sales constraints $x_s^i \geq c_s$ for every i, s, with $c = (c_1, c_2, \ldots, c_s, \ldots) \in l^p(\pi)$. We also have for every $(x^1, x^2, \ldots, x^m)$ which belongs to $\mathcal{A}$:*

$$x_s^i = e_s - \sum_{j \neq i} x_s^j \leq e_s - \sum_{j \neq i} c_s \leq |e_s| + (m-1)|c_s|.$$

$\mathcal{A}$ is hence compact for the product topology. Suppose that $(x^1(n), x^2(n), \ldots, x^m(n)) \in \mathcal{A}$ converges to $(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^m)$ for the product topology when n tends to infinity. We have for every i,

$$c_s \leq \overline{x}_s^i \leq |e_s| + (m-1)|c_s|.$$

This implies that $(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^m) \in l^p(\pi)$. The set of individually rational allocations $\mathcal{A}$ is compact in $l^p(\pi)$. That implies the compactness of $\mathcal{U}$ and the existence of a quasi-equilibrium. This explain why in our paper, the interesting case is the one where there is no short-sales constraints.

5 Appendix

Example 1 *Consider an economy with two agents $(i = 1, 2)$, with endowments equal to 0. The probabilities are equivalent: $\pi_s^1 = \left(\frac{1}{2}\right)^s$, $\pi_s^2 = \frac{1}{S_\alpha} \frac{1-\alpha^s}{2^s}$, where $1 < \alpha < 1$, and $S_\alpha = \sum_s \frac{1-\alpha^s}{2^s}$.*

The reward utilities satisfy

$$\begin{aligned} u^{1\prime}(x) &= b^1, \ \forall x \leq 0 \\ u^{1\prime}(+\infty) &= 0 \\ u^1(0) &= 0 \\ u^{2\prime}(x) &= a^2, \ \forall x \geq 0 \\ u^{2\prime}(-\infty) &= +\infty \\ u^2(0) &= 0 \end{aligned}$$

There exists $z > 0$ with $u^{1\prime}(z) < b^1$. Let $\overline{x}_s^1 = z, \forall s$. Since $u^{2\prime}(-\infty) = +\infty$, there exists $\overline{x}_s^2 < 0$ which satisfies

$$u^{2\prime}(0) = (1 - \alpha^s) u^{2\prime}(\overline{x}_s^2)$$

One can check that

$$\lambda \pi_s^1 u^{1\prime}(\overline{x}_s^1) = \pi_s^2 u^{2\prime}(\overline{x}_s^2), \forall s$$

with $\lambda = \frac{u^{2\prime}(0)}{u^{1\prime}(z)} \times \frac{1}{S_\alpha}$. Since

$$0 = u^{1\prime}(+\infty) < u^{1\prime}(z) = u^{1\prime}(\overline{x}_s^1) < b^1$$

$$a^2 = u^{2\prime}(0) < \frac{u^{2\prime}(0)}{1-\alpha^s} = u^{2\prime}(\overline{x}_s^2) < u^{2\prime}(-\infty) = +\infty$$

*no-arbitrage condition (**NA**) is satisfied.*

We now show that no equilibrium exists. Assume there exists an equilibrium $(p,(x^1,x^2))$ *with* $x_s^1 = x_s = -x_s^2$. *We have*

$$\begin{aligned} \forall s,\ \lambda_1 \pi_s^1 u^{1\prime}(x_s) &= \lambda_2 \pi_s^2 u^{2\prime}(-x_s) \\ \text{or } \lambda \pi_s^1 u^{1\prime}(x_s) &= \pi_s^2 u^{2\prime}(-x_s),\ \text{with } \lambda = \frac{\lambda_1}{\lambda_2} \end{aligned}$$

For all s:

$$\lambda' \frac{1}{2^s} u^{1\prime}(x_s) = \frac{1-\alpha^s}{2^s} u^{2\prime}(-x_s)$$

or equivalently

$$\lambda' = (1-\alpha^s)\frac{u^{2\prime}(-x_s)}{u^{1\prime}(x_s)}, \forall s$$

with $\lambda' = \lambda S_\alpha$. *Since* $\sum_s p_s x_s = 0$ *and* $p_s > 0$ *for any s, one must have* s_0 *with* $x_{s_0} \leq 0$. *In this case*

$$\lambda' = (1-\alpha^{s_0})\frac{a^2}{b^1}$$

and for any $s \neq s_0$, $x_s > 0$. *We then obtain*

$$\frac{u^{2\prime}(-x_{s_0+1})}{u^{1\prime}(x_{s_0+1})} > \frac{u^{2\prime}(0)}{u^{1\prime}(0)} = \frac{a^2}{b^1}$$

since $x_{s_0+1} > 0$. *Now, because* $1-\alpha^{s_0+1} > 1-\alpha^{s_0}$ *we obtain, on the one one hand:*

$$\begin{aligned} \lambda' &= (1-\alpha^{s_0+1})\frac{u^{2\prime}(-x_{s_0+1})}{u^{1\prime}(x_{s_0+1})} \\ &> (1-\alpha^{s_0})\frac{u^{2\prime}(-x_{s_0+1})}{u^{1\prime}(x_{s_0+1})} \\ &> (1-\alpha^{s_0})\frac{a^2}{b^1} \end{aligned}$$

and on the other hand

$$\lambda' = (1-\alpha^{s_0})\frac{a^2}{b^1}$$

which is a contradiction. Then there exists no equilibrium.

Example 2 *Consider an economy with two agents* $(i = 1, 2)$, *with endowments equal to* 0. *The probabilities are equivalent:* $\pi_s^1 = \left(\frac{1}{2}\right)^s$, $\pi_s^2 = \frac{1}{S_\alpha}\frac{1-\alpha^s}{2^s}$, *where* $1 < \alpha < 1$, *and* $S_\alpha = \sum_s \frac{1-\alpha^s}{2^s}$. *Agent* 1 *is risk neutral.*

The reward utilities satisfy

$$\begin{aligned} u^{1\prime}(x) &= 1,\ \forall x \in \mathbb{R} \\ u^{2\prime}(x) &= a^2,\ \forall x \geq 0 \\ u^{2\prime}(-\infty) &= +\infty \\ u^{2\prime}(x) &> a^2\ \forall x < 0. \end{aligned}$$

Assume there exists an equilibrium $(p,(x^1,x^2))$. *Then* $x_s^1 = -x_s^2 = x_s$ *for any* s. *There exists* $\lambda > 0$ *such that*

$$\begin{aligned} \frac{\lambda}{2^s} u^{1\prime}(x_s) &= \frac{1-\alpha^s}{S_\alpha 2^s} u^{2\prime}(-x_s) \forall s \\ \Leftrightarrow u^{2\prime}(-x_s) &= \lambda \frac{S_\alpha}{1-\alpha^s} \forall s. \end{aligned}$$

Since $\sum_s p_s x_s = 0$ *and* $p_s > 0$ *for all* s, *there exists* $x_{s_0} \leq 0$, *i.e* $-x_{s_0} \geq 0$. *Then*

$$a^2 = \lambda \frac{S_\alpha}{1-\alpha^{s_0}}$$

and $\forall s \neq s_0$, $x_s > 0$. *Hence* $u^{2\prime}(-x_{s_0+1}) > a^2$. *This implies*

$$\lambda \frac{S_\alpha}{1-\alpha^{s_0+1}} > \lambda \frac{S_\alpha}{1-\alpha^{s_0}} \Rightarrow \alpha^{s_0+1} > \alpha^{s_0}.$$

A contradiction.

Example 3 *Agent* 2 *is risk neutral. The reward utilities satisfy*

$$\begin{aligned} u^{1\prime}(x) &= b^1, \ \forall x \leq 0 \\ u^{1\prime}(+\infty) &= 0 \\ u^{1\prime}(x) &< b^1 \ \forall x > 0 \\ u^{2\prime}(x) &= 1 \ \forall x \in \mathbb{R}. \end{aligned}$$

Assume there exists an equilibrium $(p,(x^1,x^2))$. *Then* $x_s^1 = -x_s^2 = x_s$ *for any* s. *There exists* $\lambda > 0$ *such that*

$$u^{1\prime}(x_s) = \lambda \frac{1-\alpha^s}{S_\alpha} \forall s.$$

Since $\sum_s p_s x_s = 0$ *and* $p_s > 0$ *for all* s, *there exists* $x_{s_0} \leq 0$, *i.e* $-x_{s_0} \geq 0$. *Then*

$$b^1 = \lambda \frac{1-\alpha^{s_0}}{S_\alpha}$$

and $\forall s \neq s_0$, $x_s > 0$. *Hence* $u^{1\prime}(x_{s_0+1}) < b^1$. *This implies*

$$\lambda \frac{1-\alpha^{s_0+1}}{S_\alpha} < \lambda \frac{1-\alpha^{s_0}}{S_\alpha} \Rightarrow \alpha^{s_0+1} > \alpha^{s_0}.$$

A contradiction.

The following Lemma is required for the proofs of propositions 1, 2, 3, 4. It basically shows that if (**NA**) is satisfied then, on the one hand, the projection $\mathcal{A}^i$ of $\mathcal{A}$ on the $i^{th}-$ component is bounded for any $i \in I_1$ and, on the other hand, agents in I_2 have the same belief.

Lemma 1 *Assume (**NA**).*

(i) For all $i, j \in I_2$ we have $\pi^i = \pi^j$.

(ii) Denote by π^I the common belief probability of agents in I_2. There exists a constant $C > 0$ which depends only on $\overline{p}, \overline{x}, \overline{e}$ such that for any $(x^1, \ldots, x^m) \in \mathcal{A}$ we have

$$\sum_{s=1}^{\infty} \pi_s^i |x_s^i| \leq C \text{ for all } i \in I_1$$

and

$$\sum_{s=1}^{\infty} \pi_s^I |\sum_{i \in I_2} x_s^i| \leq C.$$

(iii) $\mathcal{U}$ is bounded.

Proof:

Take any **NA**-price p. There exists $(\overline{x}^i)$, $\lambda^1, \lambda^2, \ldots, \lambda^m$ positive such that for all i, s: $p_s = \lambda^i \pi_s^i u^{i\prime}(\overline{x}_s^i)$.

(i) For $i, j \in I_2$ we have $\lambda^i a^i \pi_s^i = \lambda^j a^j \pi_s^j$, $\forall s$. Take the sum we have:

$$\sum_{s=1}^{\infty} \lambda^i a^i \pi_s^i = \sum_{s=1}^{\infty} \lambda^j a^j \pi_s^j.$$

This implies $\lambda^i a^i = \lambda^j a^j$, hence $\pi^i = \pi^j$.

(ii) We firstly prove that the exists $C > 0$ such that:

$$\sum_{i \in I_1} \sum_{s=1}^{\infty} \pi_s^i |x_s^i| \leq C.$$

Define $e' = \overline{e} - \sum_{i \in I_2} x^i = \sum_{i \in I_1} x^i \in l^1(\pi)$.

For $i \in I_1$, since $a^i < \inf_s u^{i\prime}(\overline{x}_s^i) \leq \sup_s u^{i\prime}(\overline{x}_s^i) < b^i$, we have $\overline{x}^i \in l^\infty(\pi)$. Observe that $p \in l^\infty(\pi)$.

Choose $\eta > 0$ such that

$$a^i < u^{i\prime}(\overline{x}_s^i)(1+\eta) < b^i \text{ for} \tag{1}$$

for all $i \in I_1$. Then we define the price q as follows: $\forall i, j \in I_1$,

$$q_s = p_s(1+\eta) = \lambda^i \pi_s^i u^{i\prime}(\overline{x}_s^i)(1+\eta) = \lambda^j \pi_s^j u^{j\prime}(\overline{x}_s^j)(1+\eta).$$

It follows from (1) that, for each $i \in I_1$, there exist $\overline{z}^i$ such that $\forall s$, $q_s = \lambda^i \pi_s^i u^{i\prime}(\overline{z}_s^i)$. Observe that $a^i < \inf_s u^{i\prime}(z_s^i) \leq \sup_s u^{i\prime}(z_s^i) < b^i$, so $\overline{z}^i \in l^\infty(\pi)$. Observe also that $\forall s$, $p_s < q_s$.

Denote

$$x^+ \ := \begin{cases} x \ \text{ if } \ x > 0 \\ 0 \text{ if } x \leq 0 \end{cases}$$

$$x^- \ := \begin{cases} -x \ \text{ if } \ x < 0 \\ 0 \text{ if } x \geq 0 \end{cases}$$

Observe that $x = x^+ - x^-$, $|x| = x^+ + x^-$ and $u(x) = u(x^+) + u(-x^-) - u(0)$.

Now we fix $N \in \mathbb{N}$. For $i \in I_1$, from the concavity of the utility function u^i we have

$$\begin{aligned} \lambda^i \sum_{s=1}^N \pi_s^i u^i(\overline{x}_s^i) - \lambda^i \sum_{s=1}^N \pi_s^i u^i(x_s^{i+}) &\geq \lambda^i \sum_{s=1}^N \pi_s^i u^{i\prime}(\overline{x}_s^i)(\overline{x}_s^i - x_s^{i+}) \\ \lambda^i \sum_{s=1}^N \pi_s^i u^i(\overline{z}_s^i) - \lambda^i \sum_{s=1}^N \pi_s^i u^i(-x_s^{i-}) &\geq \lambda^i \sum_{s=1}^N \pi_s^i u^{i\prime}(\overline{z}_s^i)(\overline{z}_s^i + x_s^{i-}). \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda^i \sum_{s=1}^N \pi_s^i u^{i\prime}(\overline{z}_s^i) x_s^{i-} \ \leq \ & \lambda^i \sum_{s=1}^N \pi_s^i [u^i(\overline{z}_s^i) + u^i(\overline{x}_s^i) - u^i(x_s^{i+}) - u^i(-x_s^{i-})] \\ & -\lambda^i \sum_{s=1}^N \pi_s^i u^{i\prime}(\overline{z}_s^i)\overline{z}_s^i + \lambda^i \sum_{s=1}^N \pi_s^i u^{i\prime}(\overline{x}_s^i) x_s^{i+} - \lambda^i \sum_{s=1}^N \pi_s^i u^{i\prime}(\overline{x}_s^i)\overline{x}_s^i. \end{aligned}$$

Define $U_N^i(x) := \sum_{s=1}^N \pi_s^i u^i(x_s)$. Note that $\lim_{N\to\infty} U_N^i(x) = U^i(x)$. The above inequality implies

$$\begin{aligned} \sum_{s=1}^N q_s x_s^{i-} \ &\leq \ \lambda^i [U_N^i(\overline{z}^i) + U_N^i(\overline{x}^i) - U_N^i(x^i) - U_N^i(0)] \\ &- \ \sum_{s=1}^N q_s \overline{z}_s^i + \sum_{s=1}^N p_s x_s^{i+} - \sum_{s=1}^N p_s \overline{x}_s^i \\ &\leq \ \lambda^i [U_N^i(\overline{z}^i) + U_N^i(\overline{x}^i) - U_N^i(x^i) - U_N^i(0)] \\ &- \ \sum_{s=1}^N p_s^i \overline{x}_s^i - \sum_{s=1}^N q_s \overline{z}_s^i + \sum_{s=1}^N p_s x_s^{i+} \\ = \ & \ C_N^i + \sum_{s=1}^N p_s x_s^{i+} \end{aligned}$$

where $C_N^i = \lambda^i [U_N^i(\overline{z}^i) + U_N^i(\overline{x}^i) - U_N^i(x^i) - U_N^i(0)] - \sum_{s=1}^N p_s^i \overline{x}_s^i - \sum_{s=1}^N q_s \overline{z}_s^i$.

Observe that since $\overline{x}^i$ et $\overline{z}^i$ belong to $l^\infty(\pi)$, the limit $\lim_N C_N^i$ exists.

Hence, $\forall i$

$$\sum_{s=1}^N (q_s - p_s) x_s^{i-} \leq C_N^i + \sum_{s=1}^N p_s x_s^i.$$

Thus, we have

$$\sum_{i\in I_1}\sum_{s=1}^{N}(q_s-p_s)x_s^{i-} \le \sum_{i\in I_1} C_N^i + \sum_{i\in I_1}\sum_{s=1}^{N} p_s x_s^i = \sum_{i\in I_1} C_N^i + \sum_{s=1}^{N} p_s e'_s.$$

Since $e' \in l^1(\pi)$, $\sum_{i\in I_1} C_N^i + \sum_{s=1}^{N} p_s e'_s$ converges. Now let N tends to infinity. Notice that $U_N^i(x) \to U^i(x)$ for all x, and recall that $U^i(x^i) \ge U^i(e^i)$, with $\overline{x}^i, \overline{z}^i \in l^\infty(\pi)$. We then have

$$\limsup_{N\to\infty} C_N^i \le \lambda^i[U^i(\overline{z}^i) + U^i(\overline{x}^i) - U^i(e^i) - U^i(0)] - \sum_{s=1}^{\infty} q_s \overline{z}_s^i - \sum_{s=1}^{\infty} p_s \overline{x}_s^i =: C^i.$$

Thus,

$$\sum_{i\in I_1}\sum_{s=1}^{\infty}(q_s-p_s)x_s^{i-} \le \sum_{i\in I_1} C^i + \sum_{s=1}^{\infty} p_s e'_s =: C_1 + \sum_{s=1}^{\infty} p_s e'_s.$$

We also have

$$\sum_{i\in I_1}\sum_{s=1}^{\infty}(q_s-p_s)(x_s^{i+} - x_s^{i-}) = \sum_{i\in I_1}\sum_{s=1}^{\infty}(q_s-p_s)x_s^i = \sum_{s=1}^{\infty}(q_s-p_s)e'_s$$

which implies

$$\begin{aligned}
\sum_{i\in I_1}\sum_{s=1}^{\infty}(q_s-p_s)x_s^{i+} &= \sum_{s=1}^{\infty}(q_s-p_s)e'_s + \sum_{i\in I_1}\sum_{s=1}^{\infty}(q_s-p_s)x_s^{i-} \\
&\le C_1 + \sum_{s=1}^{\infty} p_s e'_s + \sum_{s=1}^{\infty}(q_s-p_s)e'_s \\
&= C_1 + \sum_{s=1}^{\infty} q_s e'_s.
\end{aligned}$$

Thus for $i \in I_1$

$$\begin{aligned}
\sum_{s=1}^{\infty}(q_s-p_s)|x_s^i| &\le 2C_1 + \sum_{s=1}^{\infty}(p_s+q_s)e'_s \\
&= 2C_1 + (2+\eta)\sum_{s=1}^{\infty} p_s e'_s \\
&= 2C_1 + (2+\eta)\sum_{s=1}^{\infty} p_s \overline{e}_s - (2+\eta)\sum_{i\in I_2}\sum_{s=1}^{\infty} p_s x_s^i \\
&= 2C_1 + (2+\eta)\sum_{s=1}^{\infty} p_s \overline{e}_s - (2+\eta)\sum_{i\in I_2} \lambda^i a^i \sum_{s=1}^{\infty} \pi_s^i x_s^i \\
&= 2C_1 + (2+\eta)\sum_{s=1}^{\infty} p_s \overline{e}_s - (2+\eta)\sum_{i\in I_2} \lambda^i U^i(x^i) \\
&\le 2C_1 + (2+\eta)\sum_{s=1}^{\infty} p_s \overline{e}_s - (2+\eta)\sum_{i\in I_2} \lambda^i U^i(e^i) \\
&= C^2.
\end{aligned}$$

then

$$\eta \sum_{s=1}^{\infty} p_s |x_s^i| \leq C^2.$$

Let $\mu_i := \inf_s u^{i\prime}(\bar{x}_s^i) > 0$, and $\mu := \min_i \mu_i$. Then $\sum_{s=1}^{\infty} p_s |x_s^i| \geq \mu \sum_{s=1}^{\infty} \pi_s^i |x_s^i|$ which implies for all $i \in I_1$

$$\sum_{s=1}^{\infty} \pi_s^i |x_s^i| \leq D_1$$

with $D_1 = C^2/(\eta\mu)$.

For I_2 we have:

$$\begin{aligned} \sum_{s=1}^{\infty} \pi_s^I |\sum_{i \in I_2} x_s^i| &\leq \sum_{s=1}^{\infty} \pi_s^I |\overline{e}_s| + \sum_{i \in I_1} \sum_{s=1}^{\infty} \pi_s^i |x_s^i| \\ &\leq D_2 \end{aligned}$$

with $D_2 = \sum_{s=1}^{\infty} \pi_s^I |\overline{e}_s| + |I_1| D_1$.

We take $C = \max\{D_1, D_2\}$.

(iii) The utility set $\mathcal{U}$ is bounded by Jensen inequality. ■

Proof of Proposition 1

(i) The proof of the boundedness of $\mathcal{U}$ comes from Lemma 1.

(ii.1) Consider the case where $I_2 = \emptyset$.

(a) Assume $u^{i\prime}(-\infty) = +\infty$, for any i. Let z satisfy $a^1 < u^{1\prime}(z)$. For $i > 1$, let

$$\zeta_s^i = \frac{\pi_s^1}{\pi_s^i} u^{1\prime}(z), \forall s$$

Then $\frac{1}{h} u^{1\prime}(z) \geq \zeta_s^i \geq h u^{1\prime}(z)$. One can find λ^i s.t. $\frac{\zeta_s^i}{\lambda^i} \geq \alpha^i > a^i$. Define

$$u^{i\prime}(x_s^i) = \frac{\zeta_s^i}{\lambda^i}, \forall s$$

Then

$$a^i < \alpha^i \leq u^{i\prime}(x_s^i) \leq \frac{1}{\lambda^i h} u^{1\prime}(z), \forall s$$

Since $b^i = +\infty$, we have $x^i \in l^\infty(\pi)$. Obviously

$$\lambda^i \pi_s^i u^{i\prime}(x_s^i) = \pi_s^1 u^{1\prime}(z), \forall s.$$

(b) Assume $u^{i\prime}(+\infty) = 0$ for all i. Let z satisfy $0 < u^{1\prime}(z) < b^1$. Define ζ_s^i as before. We have $\zeta_s^i \leq \frac{1}{h} u^{1\prime}(z)$. Choose λ^i s.t.

$\frac{\zeta_s^i}{\lambda^i} \leq \beta^i < b^i$. Then define x_s^i as before. Using the same arguments, we have

$$\lambda^i \pi_s^i u^{i\prime}(x_s^i) = \pi_s^1 u^{1\prime}(z), \forall s.$$

(ii.2) Now we consider the case where $I_2 \neq \emptyset$. If (**NA**) holds, from Lemma 1,$\pi^k = \pi^l, \forall k \in I_2, \forall l \in I_2$.

Conversely, assume that $\pi^k = \pi^l = \pi', \forall k \in I_2, \forall l \in I_2$. Assume $1 \in I_2$. For $i \in I_2, i \neq 1$, choose λ^i such that $\lambda^i a^i = a^1$.
Consider the case $u^{i\prime}(-\infty) = +\infty, \forall i \in I_1$.
For $i \in I_1$, choose as before ζ_s^i:

$$\frac{1}{h}a^1 \geq \zeta_s^i = \frac{a^1\pi_s^1}{\pi_s^i} \geq a^1 h$$

There exists λ^i s.t.

$$\frac{\zeta_s^i}{\lambda^i} \geq \alpha^i > a^i$$

and

$$a^i < u^{i\prime}(x_s^i) = \frac{\zeta_s^i}{\lambda^i} \leq \frac{1}{h}a_1.$$

Since $b^i = +\infty$, we have $x^i \in l^\infty(\pi)$. Obviously

$$\lambda^i \pi_s^i u^{i\prime}(x_s^i) = \pi_s^1 u^{1\prime}(z), \forall s.$$

The same argument as before if $u^{i\prime}(+\infty) = 0, \forall i \in I_1$. ■

Lemma 2 *A closed, bounded set B in $l^1(\pi)$ is compact if and only if B satisfies the following property: For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in B$ we have*

$$\sum_{s \geq N} \pi_s |x_s| < \epsilon.$$

Proof: Suppose that B is compact and there exists a subsequence $\{x(n)\}_n$ of B, $\epsilon > 0$ such that

$$\sum_{s=n}^{\infty} \pi_s |x_s(n)| > \epsilon, \forall n.$$

Without loss of generality, we can assume that $x(n)$ converges to x in $l^1(\pi)$ or $\|x(n) - x\|_{l^1(\pi)} \to 0$.

By choosing N large enough such that $\|x(n) - x\|_{l^1(\pi)} < \frac{\epsilon}{2}$ and $\sum_{s \geq n} \pi_s |x_s| < \frac{\epsilon}{2}$ for all $n \geq N$. And for all $n \geq N$ we have

$$\begin{aligned} \sum_{s=n}^{\infty} \pi_s |x_s| &\geq \sum_{s=n}^{\infty} \pi_s |x_s(n)| - \sum_{s=n}^{\infty} \pi_s |x_s(n) - x_s| \\ &> \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}. \end{aligned}$$

A contradiction.

Now we suppose that for any $\epsilon > 0$, there exists N such that $\forall x \in B$, $\sum_{s=N}^{\infty} \pi_s |x_s| < \epsilon$.

We have to prove that for any sequence $x(n) \in B$, there exists a convergent subsequence of $x(n)$ in $l^1(\pi)$.

Since B is bounded in $l^1(\pi)$, there exists $a > 0$ such that, $\forall\, x \in B,\ \sum_{s\geq 1} \pi_s |x_s| \leq a$.

Since $\{x(n)\}_n$ belong to a compact set for the product topology, there exists a subsequence $\{x(n_k)\}_k$ which converges to $\overline{x}$ for the product topology. In particular for all s, $x_s(n_k)$ converges to $\overline{x}_s$ when $k \to \infty$.

Fix $\epsilon > 0$. We will prove that for k, l big enough, $\|x(n_k) - x(n_l)\|_{l^1(\pi)} < \epsilon$. Choose $N > 0$ such that for all $x \in B$, $\sum_{s\geq N} \pi_s |x_s| < \frac{\epsilon}{4}$. Choose M such that for all $n_k > M$ we have $\sum_{s=1}^{M} \pi_s |x_s(n_k) - \overline{x}_s| < \frac{\epsilon}{4}$. For all $n_k \geq N_0, n_l \geq N_0$ where $N_0 = \max\{N, M\}$ we have $\sum_{s=1}^{N_0} \pi_s |x_s(n_k) - x_s(n_l)| \leq \sum_{s=1}^{N_0} \pi_s |x_s(n_k) - \overline{x}_s| + \sum_{s=1}^{N_0} \pi_s |x_s(n_k) - \overline{x}_s| < \frac{\epsilon}{2}$.

Then

$$\begin{aligned} \sum_{s\geq 1} \pi_s |x_s(n_k) - x_s(n_l)| &\leq \sum_{s=1}^{N_0-1} \pi_s |x_s(n_k) - x_s(n_l)| + \sum_{s\geq N_0} \pi_s |x_s(n_k)| + \sum_{s\geq N_0} \pi_s |x_s(n_l)| \\ &< \epsilon \end{aligned}$$

Hence $\{x(n_k)\}_k$ is a Cauchy sequence, then it converges in $l^1(\pi)$ topology. So B is compact in $l^1(\pi)$ topology. ■

Corollary 1 *1. A closed, bounded set B in $l^1(\pi)$ is compact for $l^1(\pi)$-topology if and only if it is compact for the weak topology $\sigma(l^1(\pi)), l^\infty(\pi))$.*
2. A closed, bounded set B in $l^p(\pi)$, $p > 1$ is compact for $l^1(\pi)$.

Proof: 1. Since Lemma 2 is equivalent to the Dunford-Pettis criterion, the result follows.
2. For $p > 1$, a closed bounded set is $\sigma(l^p(\pi), l^q(\pi))$-compact. But it is also $\sigma(l^1(\pi), l^\infty(\pi))$-compact, since $l^p(\pi) \subset l^1(\pi)$ and $l^\infty(\pi) \subset l^q(\pi)$. Apply statement 1. ■

Proof of Proposition 2

The idea of the proof is that, if the attainable allocation sequence does not belong to a weakly compact set, then for some state s, there will be an agent i such that x_s^i tends to $+\infty$ and an agent j such that x_s^j tends to $-\infty$. Then by reducing x_s^i and increasing x_s^j, the value of $U^i(x^i)$ does not diminish very much. Because $b^j = +\infty$, the value of $U^j(x^j)$ will become very large, even tends to infinity, and that leads us to a contradiction with the bounded property of $\mathcal{U}$.

Assume the contrary: $\mathcal{A}$ is not compact. Then from Lemma 1, there exists a sequence $\{(x^1(n), x^2(n), ..., x^m(n))\}_n \subset \mathcal{A}$, an agent i and a constant $\epsilon > 0$ such that

$$\forall\, n,\ \sum_{s=n}^{\infty} \pi_s^i |x_s^i(n)| > \epsilon.$$

Denote for all k, $v^k := \limsup_{n\to\infty} U^k(x^k(n))$.

By Lemma 1, $\mathcal{A}$ is bounded in $l^1(\pi)$. We can suppose, without loss generality, that $\sum_{s=n}^{\infty} \pi_s^i |x_s^i(n)| \to c^i > 0$ when $n \to \infty$. This implies $\lim_n \sum_{s=n}^{\infty} \pi_s^i x_s^{i+}(n) - \lim_n \sum_{s=n}^{\infty} \pi_s^i x_s^{i-}(n) = c^i$. The limits of these two sums exist because $x^i \in l^1(\pi)$. We know that $\sum_{j\neq i} x_s^i(n) = \overline{e}_s - x_s^i(n)$. So, for every s, $\exists j$ such that $x_s^j(n) \leq -\frac{x_s^i(n)-|\overline{e}_s|}{m-1}$. Since there is a finite number of agents $j \neq i$, we can assume that, for simplicity, there exist i and j which satisfy two properties:

1. $\exists\ E_n^i \subset \mathbb{N} \cap \{s \geq n\}$, $x_s^i > 0$ for all $s \in E_n^i$ and

$$\lim_n \sum_{s\in E_n^i} \pi_s^i x_s^i(n) = c^i > 0.$$

2. For all $s \in E_n^i$

$$x_s^j(n) \leq -\frac{x_s^i(n) - |\overline{e}_s|}{m-1}.$$

With each $M > 0$, define the set $S_n^i \subset E_n^i$ as follows

$$S_n^i = \{s : x_s^i(n) > |\overline{e}_s| + M(m-1)\}. \tag{2}$$

We have an observation: $\lim_n \sum_{E_n^i \setminus S_n^i} \pi_s^i x_s^i(n) = 0$. Indeed

$$\begin{aligned} \lim_{n\to\infty} \sum_{s\in E_n^i\setminus S_n^i} \pi_s^i x_s^i(n) &\leq \sum_{s\in E_n^i\setminus S_n^i} \pi_s^i \left(|\overline{e}_s| + M(m-1)\right) \\ &\leq \sum_{s=n}^{\infty} \pi_s^i |\overline{e}_s| + M(m-1)\sum_{s=n}^{\infty} \pi_s^i \end{aligned}$$

which tends to zero, since $\overline{e} \in l^1(\pi)$.

Hence we have $S_n^i \neq \emptyset$ for all n big enough, and

$$\lim_{n\to\infty} \sum_{s\in S_n^i} \pi_s^i x_s^i(n) = c^i.$$

We have

$$x_s^j(n) \leq \frac{|\overline{e}_s| - x_s^i(n)}{m-1} < -M. \tag{3}$$

Since π^i and π^j are equivalent, we can assume that $\lim_n \sum_{s\in S_n^i} \pi_s^j x_s^i(n) = c^j > 0$. Notice that these limits do not depend on M.

Define $\alpha := \min(v^k, v^i - \frac{u^{i\prime}(0)c^i}{m-1}) - 1, (k = 1, \ldots, m)$. Define $\mathcal{A}_\alpha$ the set of $(x^k) \in l^1(\pi)$ satisfies $U^k(x^k) \geq \alpha\ \forall k$ and $\sum x^k = \overline{e}$. From Lemma 1 we know that there exists $C > 0$ such that $U^j(x^j) < C$ for all $(x^1, \ldots, x^m) \in \mathcal{A}_\alpha$. Notice that our sequence $(x^k(n)) \in \mathcal{A}_\alpha$ for n large enough.

Since $b^j = +\infty$ we can choose M very big such that

$$v^j + \frac{u^{j\prime}(-M)c^j}{m-1} > C.$$

Now consider the sequence $(y^1(n), y^2(n), \ldots, y^m(n))$ defined as follows

$$y_s^i(n) := x_s^i(n) - \frac{x_s^i - |\overline{e}_s|}{m-1} + M \text{ with } s \in S_n^i,$$

$$y_s^j(n) := x_s^j(n) + \frac{x_s^i - |\overline{e}_s|}{m-1} - M \text{ with } s \in S_n^i.$$

Let $y_s^k = x_s^k$ with every $k \neq i, j$ or $s \notin S_n^i$.

Notice that $\sum_i y^i(n) = \overline{e}$, and $y_s^i(n) \leq x_s^i(n)$, $y_s^j(n) \geq x_s^j(n)$ for all s. We will prove that $\{U^l(y^l(n))\}_{l=1,m}$ is bounded below by α, but $U^j(y^j(n))$ is not bounded above by C. And this is a contradiction.

Indeed,

$$\begin{aligned}
U^i(y^i(n)) - U^i(x^i(n)) &= \sum_{s\in S_n^i} \pi_s^i(u^i(y_s^i(n)) - u^i(x_s^i(n))) \\
&\geq \sum_{s\in S_n^i} \pi_s^i u^{i\prime}(x_s^i(n) - \frac{x_s^i(n) - |\overline{e}_s|}{m-1} + M)(-\frac{x_s^i(n) - |\overline{e}_s|}{m-1} + M) \\
&\geq \sum_{s\in S_n^i} \pi_s^i u^{i\prime}(M)(-\frac{x_s^i(n)}{m-1}) + u^{i\prime}(M)(\frac{|\overline{e}_s|}{m-1} + M) \sum_{s\in S_n^i} \pi_s^i \\
&\geq -\frac{u^{i\prime}(M)}{m-1} \sum_{s\in S_n^i} \pi_s^i x_s^i(n) + u^{i\prime}(M)(\frac{|\overline{e}_s|}{m-1} + M) \sum_{s\in S_n^i} \pi_s^i.
\end{aligned}$$

When $n \to \infty$, the second term of the right hand side term in the inequality above tends to zero while first term tends to $-\frac{u^{i\prime}(M)c^i}{m-1}$. Thus,

$$\limsup_{n\to\infty} U^i(y^i(n)) \geq v^i - \frac{u^{i\prime}(M)c^i}{m-1} \geq v^i - \frac{u^{i\prime}(0)c^i}{m-1} > \alpha.$$

For n large enough, $U^k(y^k(n))$ is bounded below by α,$\forall k \neq j$. Then we can estimate the limit of $U^j(y^j(n))$ when $n \to \infty$,

$$\begin{aligned}
U^j(y^j(n)) - U^j(x^j(n)) &= \sum_{s\in S_n^i} \pi_s^j(u^j(y_s^j(n)) - u^j(x_s^j(n))) \\
&\geq \sum_{s\in S_n^i} \pi_s^j u^{j\prime}(x_s^j(n) + \frac{x_s^i(n) - |\overline{e}_s|}{m-1} - M)(\frac{x_s^i(n) - |\overline{e}_s|}{m-1} - M)
\end{aligned}$$

$$\begin{aligned}
U^j(y^j(n)) - U^j(x^j(n)) &\geq \sum_{s\in S_n^i} \pi_s^j u^{j\prime}(-M)(\frac{x_s^i(n) - |\overline{e}_s|}{m-1} - M) \\
&\geq \frac{u^{j\prime}(-M)}{m-1} \sum_{s\in S_n^i} \pi_s^j x_s^i(n) - M u^{j\prime}(-M) \sum_{s\in S_n^i} \pi_s^j - \frac{u^{j\prime}(-M)}{m-1} \sum_{s\in S_n^i} |\overline{e}_s| \pi_s^j.
\end{aligned}$$

Take the limit

$$\limsup_{n\to\infty} U^j(y^j(n)) \geq v^j + \frac{u^{j\prime}(-M)c^j}{m-1} > C.$$

A contradiction. Hence $\mathcal{A}$ is $l^1(\pi)$-compact. ■

For the proof of Proposition 3, we require the following lemma

Lemma 3 *Suppose that $\mathcal{A}$ is bounded and $(v^1, v^2, \ldots, v^m)$ is in the closure of $\mathcal{U}$. Suppose that there exists a sequence $\{x(n)\}_n \subset \mathcal{A}$ such that there exists i such that $\lim_n U^i(x^i(n)) > v^i$, and for all $j \neq i$, $\lim_n U^j(x^j(n)) \geq v^j$. Then $(v^1, v^2, \ldots, v^m) \in \mathcal{U}$.*

Proof: Fix $t \in \mathbb{N}$ arbitrarily. Let $C > 0$ be the upper bound of $\mathcal{A}$ in $l^1(\pi)$, we know that $|x_t^j(n)| < \frac{C}{\pi_t^j}$ for all j and all n. Fix some j. We define the sequence $\{y^k(n)\}_{k=1,\ldots,m}$ as follows

$$\begin{aligned} y^k(n) &= x^k(n) \text{ if } k \neq i, j \\ y_s^i(n) &= x_s^i(n) \text{ if } s \neq t \\ y_t^i(n) &= x_t^i(n) - \epsilon \\ y_t^j(n) &= x_t^j(n) + \epsilon \end{aligned}$$

For $k \neq i, j$, $\lim_n U^k(y^k(n)) = v^k$. And we have

$$\begin{aligned} U^i(y^i(n)) - U^i(x^i(n)) &= \pi_t^i(u^i(y_t^i(n)) - u^i(x_t^i(n))) \\ &\geq \pi_t^i(-\epsilon)u^{i\prime}(x_t^i(n) - \epsilon) \geq -\epsilon\pi_t^i u^{i\prime}(-\frac{C}{\pi_t^i} - \epsilon) \end{aligned}$$

and

$$\begin{aligned} U^j(y^j(n)) - U^j(x^j(n)) &= \pi_t^j(u^j(y_t^j(n)) - u^j(x_t^j(n))) \\ &\geq \pi_t^j \epsilon u^{j\prime}(x_t^j(n) + \epsilon) \geq \epsilon\pi_t^j u^{j\prime}(\frac{C}{\pi_t^j} + \epsilon). \end{aligned}$$

Since $\liminf_n U^i(x^i(n)) > v^i$, by choosing ϵ small enough, the sequence $\{y(n)\}_n$ will satisfy $\liminf_n U^i(y^i(n)) > v^i$ and $\liminf_n U^j(y^j(n)) > v^j$.

By induction we can find a sequence $\{z^k(n)\}_n \subset \mathcal{A}$ which satisfies $\lim_n U^k(z^k(n)) > v^k$ for all $k = 1, 2, \ldots, m$. Hence $(v^1, v^2, \ldots, v^m) \in \mathcal{U}$. ■

Proof of Proposition 3

Since the (**NA**) condition holds, from Proposition 1, we know that $\mathcal{U}$ is bounded. We will prove that $\mathcal{U}$ is closed. Suppose that $(v^1, \ldots, v^m)$ belong to the closure of $\mathcal{U}$ and the sequence $\{x(n)\}_n \subset \mathcal{A}$ such that $\lim_n U^i(x^i(n)) = v^i$.

If the sequence $\{x(n)\}_n$ belongs to a compact set of $l^1(\pi)$, without loss of generality, we can suppose that $\lim_n x^i(n) = x^i$ in this topology. Since U^i is continuous, we have $U^i(x^i) \geq v^i$ for all i. Thus $(v^1, \ldots, v^m) \in \mathcal{U}$.

If the sequence $\{x(n)\}_n$ does not belong to a compact set, we can suppose that there exists $c > 0$ such that for an agent i

$$\lim_{n\to\infty} \sum_{s=n}^{\infty} \pi_s^i |x_s^i(n)| = c.$$

As in the proof of Proposition 2, we can choose a pair (i, j) which satisfies the two properties:

1. $\exists\ E_n^i \subset \mathbb{N} \cap \{s \geq n\}$, $x_s^i > 0$ for all $s \in E_n^i$ and

$$\lim_n \sum_{s\in E_n^i} \pi_s^i x_s^i(n) = c^i > 0.$$

2. For all $s \in E_n^i$

$$x_s^j(n) \leq -\frac{x_s^i(n) - |\overline{e}_s|}{m-1}.$$

With each $M > 0$, define the set $S_n^i \subset E_n^i$ as follows

$$S_n^i = \{s : x_s^i(n) > |\overline{e}_s| + M(m-1)\}. \tag{4}$$

We have an observation: $\lim_n \sum_{E_n^i \setminus S_n^i} \pi_s^i x_s^i(n) = 0$. Indeed

$$\begin{aligned} \lim_{n\to\infty} \sum_{s\in E_n^i\setminus S_n^i} \pi_s^i x_s^i(n) &\leq \sum_{s\in E_n^i\setminus S_n^i} \pi_s^i \left(|\overline{e}_s| + M(m-1)\right) \\ &\leq \sum_{s=n}^{\infty} \pi_s^i |\overline{e}_s| + M(m-1) \sum_{s=n}^{\infty} \pi_s^i \end{aligned}$$

which tends to zero, since $\overline{e} \in l^1(\pi)$.

Hence we have $S_n^i \neq \emptyset$ for all n big enough, and

$$\lim_{n\to\infty} \sum_{s\in S_n^i} \pi_s^i x_s^i(n) = c^i.$$

We have

$$x_s^j(n) \leq \frac{|\overline{e}_s| - x_s^i(n)}{m-1} < -M. \tag{5}$$

Since π^i and π^j are equivalent, we can assume that $\lim_n \sum_{s\in S_n^i} \pi_s^j x_s^i(n) = c^j > 0$. Notice that these limits do not depend on M.

Define $\alpha := \min(v^k, v^i - \frac{u^{i\prime}(0)c^i}{m-1}) - 1, (k = 1, \ldots, m)$. Define $\mathcal{A}_\alpha$ the set of $(x^k) \in l^1(\pi)$ satisfies $U^k(x^k) \geq \alpha\ \forall k$ and $\sum x^k = \overline{e}$. From Lemma 1 we know that there exists $C > 0$ such that $U^j(x^j) < C$ for all $(x^1, \ldots, x^m) \in \mathcal{A}_\alpha$. Notice that our sequence $(x^k(n)) \in \mathcal{A}_\alpha$ for n large enough. Fix $\epsilon > 0$. Since $u^{i\prime}(+\infty) = 0$ we can choose $M > 0$ such that $u^{i\prime}(M) < (m-1)\epsilon/c$. By similar arguments as in the proof of Proposition 2, we can construct the sequence $(y^k(n))$ such that:

$$\liminf_{n\to\infty} U^i(y^i(n)) \geq v^i - \frac{u^{i\prime}(M)c^i}{m-1}$$

$$\liminf_{n\to\infty} U^j(y^j(n)) \geq v^j + \frac{u^{j\prime}(-M)c^j}{m-1}$$
$$\liminf_{n\to\infty} U^k(y^k(n)) = v^k \text{ for all } k \neq i, j$$

with $c^i, c^j > 0$ and $c^i < c$ and c^i and c^j do not depend on M.

So, for n large enough, $U^i(y^i(n)) > v^i - \epsilon$, and for all $k \neq i, j$, $U^k(y^k(n)) > v^k - \epsilon$ whereas $\lim_n U^j(y^j(n)) = v^j + \frac{u^{j\prime}(-M)c^j}{m-1} > v^j + \frac{u^{j\prime}(0)c^j}{m-1} > v^j$. Let $\epsilon \to 0$ and by applying the Lemma 3, we have $(v^1, v^2, \ldots, v^m) \in \mathcal{U}$. ■

Before the proof of Theorem 3, we give here a intermediaire result, the Proposition 4.

Proposition 4 *Assume **A0**. Assume that there is only one risk neutral agent, $I_2 = \{i_0\}$. If $a^i = 0$ and $b^i = +\infty$ for all $i \in I_1$, then (**NA**) holds, and $\mathcal{U}$ is compact.*

Proof of Proposition 4

The (**NA**) condition is a direct consequen of Theorem 1. Now we prove that $\mathcal{U}$ is compact.

Suppose that the feasible sequence $\{x_n\}$ satisfies $\lim_n U^i(x_n^i) = v^i$ for every i. We prove that $(v^1, \ldots, v^m) \in \mathcal{U}$.

First case, the sequence $\{x_n\}$ belongs to a compact set, we have $(v^1, v^2, \ldots, v^m)$ belongs to $\mathcal{U}$.

Second case, the sequence $\{x_n\}$ does not belong to a compact set. By using the same arguments in the proof of Proposition 2, there exist i and j which satisfy two properties:

1. $\exists\ E_n^i \subset \mathbb{N} \cap \{s \geq n\}$, $x_s^i > 0$ for all $s \in E_n^i$ and
$$\lim_n \sum_{s\in E_n^i} \pi_s^i x_s^i(n) = c^i > 0.$$

2. For all $s \in E_n^i$
$$x_s^j(n) \leq -\frac{x_s^i(n) - |\overline{e}_s|}{m-1}.$$

If $j \neq i_0$, since $b^j = +\infty$, by using the same arguments in the proof of Proposition 2 we have a contradiction.

This implies $j = i_0$. Hence $i \neq i_0$, and we have $a^i = 0$. By using the same arguments in the proof of Proposition 3, we have $(v^1, \ldots, v^m) \in \mathcal{U}$.

Hence $\mathcal{U}$ is compact.

Proof of Theorem 3

Claim Let $f^1, \ldots, f^n$, be n vectors in $l^p(\pi)$, and $p \in l^\infty(\pi)$. Take any $x \in l^p(\pi)$ such that $p \cdot x = p \cdot \sum_i f^i$. Then there exists $x^1, \ldots, x^n$ in $l^p(\pi)$ such that $\sum_i x^i = x$ and $p \cdot x^i = p \cdot f^i$ for all i.

Proof of the claim. This is true for $n = 1$. Suppose that the claim is true for $n-1$. Take any x^1 such that $p \cdot x^1 = p \cdot f^1$. We have $p \cdot (x - x^1) = p \cdot \sum_{i=2}^n f^i$. Using the hypothesis of induction, there exists $x^2, \ldots, x^n$ such that $\sum_{i=2}^n x^i = x - x^1$ and $p \cdot x^i = p \cdot f^i$ for all i. ■

In the proof of Proposition 2, we have $\pi^i = \pi^j = \pi^I$ and $\lambda_i a^i = \lambda_j a^j = \zeta$, for all $i, j \in I_2$. For $x^I \in l^1(\pi)$ satisfying $x^I = \sum_{i \in I_2} x^i$, with $x^i \in l^1(\pi)$, define

$$U^I(x^I) = \sum_{i \in I_2} \lambda_i U^i(x^i).$$

We consider now the economy $\mathcal{E}^I$ with $|I_1| + 1$ agents, $|I_1|$ agents who are risk averse, with endowment e^i, utility function U^i, and the last agent (denoted by I) with endowment $e^I = \sum_{i \in I_2} e^i$, utility function U^I. It is easy to verify that agent I is risk neutral, with

$$U^I(x_s^I) = \zeta \sum_{s=1}^{\infty} \pi_s^I x_s^I,$$

and the new economy satisfies (**NA**) condition. By the Proposition 4, this economy has a compact rational utility set and have an equilibrium, denote by (p^*, x^*). For all $i \in I_1$, x^{*i} is the solution to

$$\begin{aligned} &\max \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i) \\ \text{s.t. } &\sum_{s=1}^{\infty} p_s^* x_s^i = \sum_{s=1}^{\infty} p_s^* e_s^i. \end{aligned}$$

and x^{*I} is the solution to

$$\begin{aligned} &\max U^I(x^I) \\ \text{s.t. } &\sum_{s=1}^{\infty} p_s^* x_s^I = \sum_{s=1}^{\infty} p_s^* e_s^I. \end{aligned}$$

If $U^I(x^i) > U^I(x^{*I})$, then $p^* \cdot x^i > p^* \cdot e^I$.

By the same arguments in the proof of the Theorem 1, there exists $\lambda_i^* > 0$, $\lambda_I > 0$ such that for $i \in I_1$, $p_s^* = \lambda_i \pi_s^i u^{i\prime}(x_s^{*i})$. For I, we have $p_s^* = \zeta \pi_s^I = \lambda_i a^i \pi_s^I$, $\forall i \in I_2$, $\forall s$. The function U^I is strictly increasing, so

$$p^* \cdot x^{*I} = p^* \cdot e^{*I} = p^* \cdot \sum_{i \in I_2} e^i.$$

By the claim, for all $i \in I_2$ there exist $x^{*i} \in l^p(\pi)$ such that $\sum_{i\in I_2} x^{*i} = x^{*I}$, and $p^* \cdot x^{*i} = p^* \cdot e^i$, $\forall i$.

Fix $i \in I_2$. Take x^i such that $U^i(x^i) > U^i(x^{*i})$. We prove that $p^* \cdot x^i > p \cdot e^i$. Indeed, we have

$$U^i(x^i) = \sum_{s=1}^{\infty} a^i \pi_s^i x_s^i = \frac{1}{\lambda_i} \sum_{s=1}^{\infty} \lambda_i a^i \pi_s^i = \frac{1}{\lambda_i} \sum_{s=1}^{\infty} p_s^* x_s^i.$$

Hence $U^i(x^i) > U^i(x^{*i})$ implies $p^* \cdot x^i > p^* \cdot x^{*i} = p^* \cdot e^i$.

We have proved that $(p^*, (x^{*i})_i)$ is an equilibrium of the model.

Suppose that the model has an equilibrium, then by using Proposition 1, we have (**NA**). The proof the Theorem 3 is completed. ■

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