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Dynamic Portfolio Insurance Strategies: Risk Management under Johnson Distributions

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Abstract

The purpose of this paper is to analyze the gap risk of dynamic portfolio insurance strategies which generalize the "Constant Proportion Portfolio Insurance" (CPPI) method by allowing the multiple to vary. We illustrate our theoretical results for conditional CPPI strategies indexed on hedge funds. For this purpose, we provide accurate estimations of hedge funds returns by means of Johnson distributions. We introduce also an EGARCH type model with Johnson innovations to describe dynamics of risky logreturns. We use both VaR and Expected Shortfall as downside risk measures to control gap risk. We provide accurate upper bounds on the multiple in order to limit this gap risk. We illustrate our theoretical results on Credit Suisse Hedge Fund Index. The time period of the analysis lies between December 1994 and December 2013.

Keywords: Portfolio insurance; CPPI; Hedge funds; Johnson distribution, gap risk, VaR, CVaR.

JEL classification: C6, G11, G24, L10.

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1 Introduction

Alternative investments have enjoyed increasing success with institutional and private investors around the world. This phenomenon of globalization has been fostered by poor market conditions of the 2000s which have illustrated the advantages of this style of portfolio management compared to traditional management. Providing returns that are decorrelated with standard market indices, hedge funds have become attractive for many fund managers. Hedge fund performances can potentially provide benefits to individuals who, either invest directly in these funds, either invest in portfolios of hedge funds managed by portfolio insurance methods (as recently illustrated).

The goal of portfolio insurance is first to limit downside risk in bearish financial markets, second to allow some participation in bullish markets. There exist several methods of portfolio insurance: OBPI (Option Based Portfolio Insurance), CPPI (Constant Proportion Portfolio Insurance), Stop-loss, ... (see for example Poncet and Portait (1997) and Bertrand and Prigent (2011) for a review about these methods). Here, we examine the CPPI method introduced by Black and Jones (1987) for equity instruments and Perold (1986), Black and Perold (1988) for fixed-income instruments (see also Black and Rouhani (1987), Roman, Koprash and Hakanoglu (1989), Black and Perold (1992)). The CPPI method uses a simplified strategy to allocate assets dynamically over time. Basically, two assets are involved: the riskless asset, B , with a constant interest rate r (usually Treasury bills or other liquid money market instruments) and the risky one, S .

Usually, the investor, with initial amount to invest V_0 , wants to recover a fixed percentage p of his initial investment at a given horizon T . To obtain a terminal portfolio value V_T higher than the insured amount pV_0 , the portfolio manager keeps the portfolio value V_t above the floor $F_t = p.V_0.e^{-r(T-t)}$ at any time t in the period $[0, T]$. For this purpose, the amount e_t (called the exposure) invested in the risky asset is equal to a fixed proportion m (called the multiple) of the excess of the portfolio value over the floor, also called the cushion and thus defined by $C_t = V_t - F_t$. Therefore, this insurance method consists in keeping C_t positive at any time t in the period. The remaining funds are invested in the riskless asset B_t .

Advantages of this portfolio insurance strategy over other approaches are its simplicity and its flexibility. Initial cushion, floor and tolerance can be determined according to the own investor's objective (see Poncet and Portait, 1997). Issuers (banks or insurance companies) may bear market risks on the insured portfolios, due to sudden large market drops. For instance, in the case of the CPPI method, issuers must at least provision the difference with their own capital if the value of the portfolio becomes smaller than the floor. Therefore, they can use, for example, stress testing to estimate their potential losses.

Thus, the issuer faces one crucial question: what exposure to the risky asset or, equivalently, what level of the multiple to accept? On one hand, since

portfolio return expectation is increasing with respect to the multiple, customers want to benefit from a multiple as high as possible. On the other hand, due to market imperfections,¹ portfolio managers must impose an upper bound on the multiple. For example, if the portfolio manager anticipates that the maximal daily historical drop (e.g. -20 %) will occur during the period, he chooses $m \leq 5$ which leads to low return expectation. Alternatively, he may consider that the maximal daily drop during the management period will never be higher than a given value (e.g. -10%). A straightforward implication is to choose m according to this new extreme value (e.g. $m \leq 10$). Another possibility is to take account of the occurrence probabilities of extreme events in the risky asset returns. Then, the portfolio manager can adopt a quantile hedging strategy by determinating the maximum multiple so that the portfolio value will always be above the floor at a given probability level (typically 99%).

This latter management approach has important practical implications. The determination of the multiple according to quantile conditions have been examined for instance in Prigent (2001) in the case of a Lévy process and in Bertrand and Prigent (2002), using extreme value theory. Hamidi *et al* (2009a,b) and Ben Ameur and Prigent (2014) also consider a modified quantile hedging strategy and show how a conditional multiple can be determined. For this purpose, Hamidi *et al* (2014) introduce quantile regressions, like for example Engel and Manganelli (2004) who study Conditional Auto Regressive Value-at-Risk by Regression Quantile (CAViaR).

In this paper, we show how the portfolio manager can choose the value of the multiple according to hedge funds return fluctuations. This allows the introduction of a conditional multiple. To illustrate our gap risk management approach, we consider both quantile and expected shortfall conditions. We consider a quite general Johnson distribution to model hedge funds returns. Our results prove that, in the generalized Johnson framework, a conditional multiple can be determined as functions of state variables such as the past returns and volatilities.

The paper is organized as follows. Section 2 introduces the modelling of hedge funds returns by Johnson distributions. Section 3 presents the basic properties of the CPPI model in this framework. Section 4 details the modified CPPI method based on a conditional multiple that is determined from quantile and expected shortfall conditions. In particular, upper bounds on the multiple are provided. Section 5 provides the empirical analysis. Some proofs are gathered in Appendix.

¹For example, it is well known that portfolio managers cannot actually modify their portfolio weights in continuous time. Additionally, asset liquidity problems may occur, especially during stock markets crashes (see Longin, 1997).

2 Modelling hedge funds returns

Many studies about hedge fund returns have shown that the mean-variance approach is not adequate to evaluate the risks and performances of these specific funds. Indeed, hedge funds returns distributions can have negative skewness and excess kurtosis, as documented by Fung and Hsieh (1997), Ackerman et al. (1999), Brown et al. (1999), Brooks and Kat (2001), Caglayan and Edwards (2001), Bacmann and Scholz (2003), Agarwal and Naik (2004)... This leads to conclude that, for this type of fund, the challenge lies not only in the first two moments but also in higher moments. In this framework, the selection of a family of distributions with sufficient flexibility to take account of the asymmetry and of the kurtosis is necessary to study the behavior of hedge funds returns. The Johnson distribution system (1949) satisfies this requirement.

2.1 Families of Johnson distributions

2.1.1 Johnson system

The Johnson system covers three main families of probability distributions that can model a wide variety of empirical distributions. Additionally, we can introduce a quite general family of probability distributions that we call the Generalized Johnson Distributions (GJD) defined as follows. Given a continuous random variable X , X has a Johnson distribution if there exists a monotonic and invertible function g such that:

$$\varepsilon = \gamma + \delta g\left(\frac{X - \xi}{\lambda}\right), \quad (1)$$

where ε is a standard normal random variable, and $\gamma, \delta, \xi, \lambda$ are four given parameters. Usually, γ and δ correspond to trend parameters, ξ is a positional parameter and finally λ corresponds to a scaling parameter.

Notation 1 *We introduce the function h and its inverse by setting:*

$$\begin{aligned} h(x) &= \gamma + \delta g\left(\frac{x - \xi}{\lambda}\right), \\ h^{-1}(z) &= \lambda g^{-1}\left(\frac{z - \gamma}{\delta}\right) + \xi. \end{aligned}$$

and also $y = \left(\frac{x - \xi}{\lambda}\right)$.

The three standard types of Johnson distribution correspond to a function $g(\cdot)$ defined by:

- A lognormal distribution

$$S_L : g(y) = \ln(y).$$

- An unbounded distribution

$$S_U : g(y) = \ln(y + \sqrt{1 + y^2}) = \sinh^{-1}(y). \quad (2)$$

- A bounded distribution

$$S_B : g(y) = \ln\left(\frac{y}{1 - y}\right).$$

The S_U distribution is unbounded in both directions while the S_B distribution is bounded in both directions $(\xi, \xi + \lambda)$. The Johnson probability density function (pdf) f_X is easily deduced from standard normal distribution. It is given by canonical transformations of the standard Gauss distribution:

$$f_X(x) =$$

$$\frac{\delta}{\lambda\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left\{\gamma + \delta g\left(\frac{x - \xi}{\lambda}\right)\right\}^2\right] \left|g'\left(\frac{x - \xi}{\lambda}\right)\right|.$$

The pdf $f_{\tilde{X}}$ corresponding to each of the three main Johnson distributions is also easily deduced from previous general form:

$$S_L : f_X(x) =$$

$$\frac{\delta}{\lambda\sqrt{2\pi}\left(\frac{x - \xi}{\lambda}\right)} \exp\left[-\frac{1}{2}\left\{\gamma + \delta \ln\left(\frac{x - \xi}{\lambda}\right)\right\}^2\right] \text{ for } \frac{x - \xi}{\lambda} < 0.$$

$$S_U : f_X(x) =$$

$$\frac{\delta}{\lambda\sqrt{2\pi}\sqrt{\left(\frac{x - \xi}{\lambda}\right)^2 + 1}} \exp\left[-\frac{1}{2}\left\{\gamma + \delta \ln\left(\frac{x - \xi}{\lambda} + \sqrt{\left(\frac{x - \xi}{\lambda}\right)^2 + 1}\right)\right\}^2\right].$$

$$S_B : f_X(x) =$$

$$\frac{\delta}{\lambda\sqrt{2\pi}\left(\frac{x - \xi}{\lambda}\right)\left(1 - \frac{x - \xi}{\lambda}\right)} \exp\left[\frac{1}{2}\left\{\gamma + \delta \ln\left(\frac{\frac{x - \xi}{\lambda}}{1 - \left(\frac{x - \xi}{\lambda}\right)}\right)\right\}^2\right]$$

$$\text{for } 0 < \frac{x - \xi}{\lambda} < 1.$$

When using Johnson type distributions, we can:

- First search among the three families of Johnson distributions to be introduced;
- Second estimate the parameters of this distribution;
- Finally, if necessary, determine the best function g , using a non parametric approach. In that latter case, function g may no longer belong to one of the three basic families.

3 CPPI with fixed multiple and Johnson distributions

We begin by briefly recalling the main properties of the standard CPPI strategy in discrete time (constant multiple).

3.1 The financial market

Consider two basic financial assets: a riskless asset, denoted by B , and a hedge fund denoted by S . Changes in asset prices are supposed to occur at discrete times along a whole management period $[0, T]$. This assumption is in accordance to actual practice when investing on hedge funds.

The riskless asset evolves according to a deterministic rate denoted by r .

The variations of the hedge funds S between two times t_{k-1} and t_k are defined by:

$$\Delta S_{t_{k+1}} = S_{t_{k+1}} - S_{t_k}.$$

Since we search for an upper bound on the multiple m , we have to focus on the left hand side of the probability distribution of $\frac{\Delta S_{t_k}}{S_{t_{k-1}}}$.

Denote also:

$$R_{t_k} = \left(\frac{S_{t_k}}{S_{t_{k-1}}} \right) \text{ and } Z_{t_k} = \ln(R_{t_k}).$$

3.2 The CPPI portfolio

Denote by V_{t_k} the value of the portfolio at time t_k . As explained in introduction, the CPPI method is based on the following portfolio insurance conditions:

- There exists a deterministic floor F_{t_k} such that at any time t_k , the value V_{t_k} must be above the floor.
- The total amount e_{t_k} invested on the underlying asset S is equal to mC_{t_k} where the cushion C_{t_k} represents the difference $V_{t_k} - F_{t_k}$, between the portfolio value and the floor.
- The multiple m is a non negative constant.
- The remaining amount $(V_{t_k} - e_{t_k})$ is invested on the riskless asset with a deterministic rate r_{t_k} for the period $[t_{k-1}, t_k]$.

The higher the multiple m , the higher the amount invested on the risky asset. Therefore, a speculative investor would choose high values for m . Nevertheless, in that case, his portfolio is riskier and as shown in what follows, his guarantee may no longer hold. Indeed, we easily deduce that the portfolio value is solution of

$$V_{t_{k+1}} = V_{t_k} + e_{t_k} \left(\frac{\Delta S_{t_{k+1}}}{S_{t_k}} \right) + (V_{t_k} - e_{t_k})r_{t_{k+1}}, \quad (3)$$

from which, we deduce the dynamics of the cushion:

$$C_{t_{k+1}} = C_{t_k} \left[1 + m \left(\frac{\Delta S_{t_{k+1}}}{S_{t_k}} \right) + (1 - m)r_{t_{k+1}} \right]. \quad (4)$$

Since, for all times t_k , the cushion must be positive, we get finally the condition: for all $k \leq n$,

$$m \left(\frac{\Delta S_{t_{k+1}}}{S_{t_k}} \right) + (1 - m)r_{t_k} \geq -1. \quad (5)$$

In fact, since r_{t_k} is relatively small, the previous inequality yields to the following relation that gives an upper bound on the multiple:

Proposition 2 *The guarantee is satisfied at any time of the management period with a probability equal to 1 if and only if*

$$\forall k \leq n, -\frac{\Delta S_{t_{k+1}}}{S_{t_k}} \leq \frac{1}{m} \text{ or equivalently } M_n = \text{Max} \left(-\frac{\Delta S_{t_{k+1}}}{S_{t_k}} \right)_{k \leq n} \leq \frac{1}{m}, \quad (6)$$

which is equivalent to:

$$\text{Max} (1 - R_{t_k}) \leq \frac{1}{m}, \text{ and also } \text{Min} (R_{t_k}) \geq \left(1 - \frac{1}{m} \right). \quad (7)$$

Let us now deeply examine this condition when the return on the risky asset follows a generalized Johnson distribution.

Denote by ϵ_{t_k} the logarithm of the normalized return conditionally on the available information at time t_{k-1} :

$$\epsilon_{t_k} = \frac{\ln R_{t_k} - \mu_{t_k}}{\sigma_{t_k}}.$$

Thus, at any time t_{k+1} the risky asset value is given by:

$$S_{t_{k+1}} = S_{t_k} \exp [\mu_{t_k} + \sigma_{t_k} \epsilon_{t_k}].$$

Assuming that the probability distribution of the innovation process ϵ_{t_k} is a generalized Johnson distribution, we have:

$$\epsilon_{t_k} = h_{t_k}(\epsilon_{t_k}) \text{ where } h_{t_k}(x) = \gamma_{t_k} + \delta_{t_k} g \left(\frac{x - \xi_{t_k}}{\lambda_{t_k}} \right),$$

where the four parameters $\gamma_{t_k}, \delta_{t_k}, \xi_{t_k}, \lambda_{t_k}$ are deterministic. We get equivalently:

$$\epsilon_{t_k} = h_{t_k}^{-1}(\epsilon_{t_k}) \text{ where } h_{t_k}^{-1}(z) = \lambda_{t_k} g^{-1} \left(\frac{z - \gamma_{t_k}}{\delta_{t_k}} \right) + \xi_{t_k}.$$

Then we deduce a general insurance condition.

Proposition 3 *The guarantee is satisfied at any time t_k of the management period with a probability equal to 1 if and only if*

$$\text{Min}(\varepsilon_{t_k}) \geq \delta_{t_k} g \left(\frac{\left[\frac{\ln(1 - \frac{1}{m}) - \mu_{t_k}}{\sigma_{t_k}} \right] - \xi_{t_k}}{\lambda_{t_k}} \right) + \gamma_{t_k}. \quad (8)$$

Remark 4 *Since the right end point d of the common distribution of the variables $\left(-\frac{\Delta S_{t_k+1}}{S_{t_k}}\right)$ is positive, we deduce that the insurance is perfect along any period $[0, T]$ if and only if m is smaller than $\frac{1}{d}$. For example, if the maximal drop is equal to 20%, then $d = 0.2$. Thus m must be smaller than 5.*

3.3 Quantile condition with a fixed multiple

Previous condition (8) is rather stringent. Therefore, we can search for an alternative rationale condition like for example a criterion such as the Value-at-Risk. This latter approach is based on quantile hedging as emphasized by Föllmer and Leukert (1999).

In that case, the new guarantee condition is that the portfolio value is always higher than the floor at a given probability $1 - a$ (for example $a = 1\%$). This yields to the following relation for a management time period $[0, T]$:

$$\mathbb{P}[C_{t_k} \geq 0, \forall t_k \in \{1, \dots, T\}] \geq 1 - a,$$

which is equivalent to:

$$\mathbb{P}[\forall t_k \in \{1, \dots, T\}, -\frac{\Delta S_{t_k+1}}{S_{t_k}} \leq \frac{1}{m}] \geq 1 - a. \quad (9)$$

In the generalized Johnson distributions framework, this leads to the following condition:

$$\mathbb{P}[\forall t_k \in \{1, \dots, T\}, \varepsilon_{t_k} \geq \delta_{t_k} g \left(\frac{\left[\frac{\ln(1 - \frac{1}{m}) - \mu_{t_k}}{\sigma_{t_k}} \right] - \xi_{t_k}}{\lambda_{t_k}} \right) + \gamma_{t_k}] \geq 1 - a. \quad (10)$$

Denote by N the cdf of the standard Gaussian distribution. If $(\varepsilon_{t_k})_k$ is i.i.d. (recall that g is increasing), then Condition (10) is equivalent to:

$$1 - N \left[\delta_{t_k} g \left(\frac{\left[\frac{\ln(1 - \frac{1}{m}) - \mu_{t_k}}{\sigma_{t_k}} \right] - \xi_{t_k}}{\lambda_{t_k}} \right) + \gamma_{t_k} \right] \geq (1 - a)^{\frac{1}{T}},$$

from which we deduce:

$$N \left[\delta_{t_k} g \left(\frac{\left[\frac{\ln(1-\frac{1}{m}) - \mu_{t_k}}{\sigma_{t_k}} \right] - \xi_{t_k}}{\lambda_{t_k}} \right) + \gamma_{t_k} \right] \leq 1 - (1-a)^{\frac{1}{T}}.$$

Therefore, for the i.i.d. case, we can characterize the quantile condition.

Proposition 5 *For small r_{t_k} values and if the sequence $(\varepsilon_{t_k})_k$ is i.i.d. with common cdf N , then we have (approximately) the following condition:*

$$\delta_{t_k} g \left(\frac{\left[\frac{\ln(1-\frac{1}{m}) - \mu_{t_k}}{\sigma_{t_k}} \right] - \xi_{t_k}}{\lambda_{t_k}} \right) + \gamma_{t_k} \leq q_{a,T}, \text{ with } q_{a,T} = N^{-1} \left[1 - (1-a)^{\frac{1}{T}} \right],$$

which provides the following upper bound on the multiple:

$$m \leq \text{Min}_k \left[\frac{1}{1 - \exp[h_{t_k}^{-1}(q_{a,T})\sigma_{t_k} + \mu_{t_k}]} \right]. \quad (11)$$

Remark 6 *Note that this upper limit on the multiple m is obviously higher than the standard limit $\frac{1}{d}$.*

For a given probability level a and for a given transformation h_{t_k} , this upper bound depends on the two state variables, namely the conditional trend μ_{t_k} and the conditional volatility σ_{t_k} .

It is an increasing function with respect to the conditional trend, while, for standard values of the probability level α ($q_{a,T} < 0$), it is a decreasing function of the conditional volatility.

As intuitively, the higher the conditional return expectation, the less stringent the quantile condition (i.e. the higher the upper bound); the higher the conditional volatility, the more stringent the quantile condition (i.e. the smaller the upper bound).

4 CPPI with a conditional multiple and Johnson distributions

In previous section, the multiple is assumed to be constant. It is chosen and fixed at initial time of the management period. In what follows, we allow the multiple to vary according to market fluctuations, in order to get sufficient flexibility to improve portfolio performance.

4.1 Conditional multiple

The CPPI method is sufficiently flexible so that several extensions can be proposed.

- For example, Estep and Kritzman (1988) have introduced the Time Invariant Portfolio Protection (TIPP), which is based on a stochastic floor to keep past profits from rises in the stock market. The new guarantee condition is as follows:

$$V_t \geq \max(P_t, k \times \sup_{s \leq t} V_s), \quad (12)$$

with k an exogenous parameter lying between 0 and 1 (usually equal to 0.9). It means that the investor does not want to loose more than a given percentage of the maximum of his past portfolio values. This portfolio strategy belongs to CPPI strategies but associated to a stochastic floor given by $\max(P_t, k \times \sup_{s \leq t} V_s)$.

- The multiple can vary according to market and portfolio dynamics. Prigent (2001) considers a more general exposure function $e_t = e(t, C_t)$ defined on $[0, T] \times \mathbb{R}^+$, positive and continuous, with conditions on $e(t, C)$ ensuring that the cushion always is positive. If the cushion is nil, then the exposure must be equal to 0 (i.e. $e(t, 0) = 0$) and if the relative sizes of jumps $\frac{\Delta S}{S}$ have a lower bound d (negative), then, for any (t, C) , we must have:

$$e(t, C) \leq -\frac{1}{d}C.$$

In what follows, we still assume a linear function of the cushion, but now the multiple is a random process depending for example on given state variables such as the risky asset volatility. We illustrate the determination of the conditional multiple as function of state variables, within the standard or generalized Johnson distributions.

4.2 Quantile hedging and conditional multiple

4.2.1 Quantile condition

In what follows, as in Ben Ameur and Prigent (2014), we focus on local quantile condition. Consider the filtration $(\mathcal{F}_{t_k})_k$ generated by the observations of the returns $(R_{t_k})_k$. Consider a given sequence of thresholds $L_{t_{k-1}} > 0$ (these thresholds can be fixed or variable). The local quantile condition is defined by:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[C_{t_k} > L_{t_{k-1}}] \geq (1 - a),$$

where $\mathbb{P}^{\mathcal{F}_{t_{k-1}}}$ denotes the conditional probability with respect to information $\mathcal{F}_{t_{k-1}}$.

Recall that the cushion value is determined from the following relation²

$$C_{t_k} = C_{t_{k-1}} \times (1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}}). \quad (13)$$

The guarantee condition at any time t_k is: $C_{t_k} > L_{t_{k-1}}$.

Case 1: ($C_{t_{k-1}} > 0$)

$$\begin{aligned} C_{t_k} > L_{t_{k-1}} &\iff 1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} > \frac{L_{t_{k-1}}}{C_{t_{k-1}}}, \\ C_{t_k} > L_{t_{k-1}} &\iff 1 + m_{t_{k-1}} \times (R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}}, \\ C_{t_k} > L_{t_{k-1}} &\iff 1 + m_{t_{k-1}} \times [\exp[h_{t_k}^{-1}(\varepsilon_{t_k})\sigma_{t_k} + \mu_{t_k}] - 1] > \frac{L_{t_{k-1}}}{C_{t_{k-1}}}. \end{aligned}$$

Case 2: ($C_{t_{k-1}} < 0$). In this case, this is not a true insurance condition. Therefore, a possible strategy is to adopt the previous condition when the cushion is positive and to invest the whole portfolio value on the riskless asset, as soon as the cushion is negative. Therefore, at any time t_{k-1} , the value of the multiple $m_{t_{k-1}}$ must be chosen such that, at time t_k , the cushion value satisfies: $C_{t_k} > L_{t_{k-1}}$. Consequently, $m_{t_{k-1}}$ can be searched as the following function:

$$m_{t_{k-1}} = m_{t_{k-1}}^{(1)} \times \mathbb{I}_{C_{t_{k-1}} > L_{t_{k-1}}} + m_{t_{k-1}}^{(2)} \times \mathbb{I}_{C_{t_{k-1}} \leq L_{t_{k-1}}}, \quad (14)$$

where both $m_{t_{k-1}}^{(1)}$ and $m_{t_{k-1}}^{(2)}$ are random variables which are $\mathcal{F}_{t_{k-1}}$ -measurable.

4.2.2 Determination of the multiple

In what follows, we search for explicit forms of the random variables $m_{t_{k-1}}^{(1)}$ and $m_{t_{k-1}}^{(2)}$. We assume that the funds dynamics corresponds to an EGARCH type model, defined by:

$$R_{t_k} = \frac{S_{t_k}}{S_{t_{k-1}}} = \exp[Z_{t_k}] \text{ with } Z_{t_k} = \mu_{t_k} + \sigma_{t_k} h_{t_k}^{-1}(\varepsilon_{t_k}),$$

²Since we consider small time intervals, we set $r_{t_k} = 0$.

where Z_{t_k} is solution of the system of the following autoregressive equations:

$$\begin{cases} Z_{t_k} = \alpha_0 + \sum_{i=1}^P \alpha_i \times Z_{t_{k-i}} + \sigma_{t_k} \times \epsilon_{t_k}, \\ \Lambda(\sigma_{t_k}) = \beta + C_0(\epsilon_{t_{k-1}}) + C_1(\epsilon_{t_{k-1}}) \times \Lambda(\sigma_{t_{k-1}}), \end{cases} \quad (15)$$

where σ_{t_k} denotes the volatility, the sequence $(\epsilon_{t_k})_k$ is i.i.d. with common pdf $f > 0$ and Λ , $C_0(\cdot)$, and $C_1(\cdot)$ are deterministic functions. The function $\Lambda : \mathbb{R}^+ \rightarrow \mathbb{R}$ is assumed to be strictly increasing. We set

$$\mu_{t_k} = \alpha_0 + \sum_{i=1}^P \alpha_i \times Z_{t_{k-i}}.$$

The information delivered by the observation of risk asset returns is generated by the $(\epsilon_{t_1}, \dots, \epsilon_{t_{k-1}})_k$. We have:

$$\mathcal{F}_{t_{k-1}} = \sigma\text{-algebra } (\epsilon_{t_1}, \dots, \epsilon_{t_{k-1}}). \quad (16)$$

Thus we deduce that the random variables $m_{t_{k-1}}^{(1)}$ and $m_{t_{k-1}}^{(2)}$ are deterministic functions of $(\epsilon_{t_1}, \dots, \epsilon_{t_{k-1}})$. Therefore, we have to search for a multiple m_{t_k} which has the following form:

$$\begin{aligned} m_{t_{k-1}} &= m^{(1)}(t_{k-1}, Z_{t_1}, \dots, Z_{t_{k-1}}, \sigma_{t_1}, \dots, \sigma_{t_k}) \times \mathbb{I}_{C_{t_{k-1}} > L_{t_{k-1}}} \\ &+ m^{(2)}(t_{k-1}, Z_{t_1}, \dots, Z_{t_{k-1}}, \sigma_{t_1}, \dots, \sigma_{t_k}) \times \mathbb{I}_{C_{t_{k-1}} \leq L_{t_{k-1}}}, \end{aligned} \quad (17)$$

where the parameter $L_{t_{k-1}}$ corresponds to a given threshold ($L_{t_{k-1}} > 0$).

Remark 7 Note that the random variables $Z_{t_{k-1}}$ are deterministic functions of the random vector $(\epsilon_{t_1}, \dots, \epsilon_{t_{k-1}})$. But, for a better financial explanation of the multiple, we explicitly introduce functions of vector $(Z_{t_1}, \dots, Z_{t_{k-1}})$ itself. This allows to analyze how the conditional multiple depends on both the volatility levels and on the past values of Z related to returns, which are here the state variables.

In order to determine the conditional multiple, two cases have to be distinguished³:

- The cushion at time t_{k-1} is non negative: $C_{t_{k-1}} > L_{t_{k-1}}$.
- The cushion at time t_{k-1} satisfies: $0 < C_{t_{k-1}} < L_{t_{k-1}}$.

Consider the conditional cdf $F^{\mathcal{F}_{t_{k-1}}}$ of $Z_{t_{k-1}}$.

We obtain the following results⁴ (the cushion $C_{t_{k-1}}$ at time t_{k-1} is assumed to be strictly positive (otherwise, we stop investing in the risky asset)).

³Since we assume that the pdf of the random variables ϵ is non negative, the probability of the event $C_{t_{k-1}} = L_{t_{k-1}}$ is null.

⁴For the proofs of the results that follow, see Appendix.

Consider at first the case where the threshold $L_{t_{k-1}}$ is arbitrary.

Proposition 8 *When $C_{t_{k-1}} > L_{t_{k-1}}$, the quantile condition at time t_{k-1} can be described as follows:*

- i) *If $\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 < 0$, then $\mathbb{P}^{\mathcal{F}_{t_{k-1}}}[C_{t_k} > L_{t_{k-1}}] = 1$.*
- ii) *If $C_{t_{k-1}} > L_{t_{k-1}}$ and $\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 > 0$, then:*
 - a) *If $\exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] < 1$, then*
If $\frac{L_{t_{k-1}}}{C_{t_{k-1}}} < \exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right]$, then $m_{t_{k-1}}$ must satisfy the following constraint:

$$m_{t_{k-1}} \leq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] - 1}. \quad (18)$$

If $\frac{L_{t_{k-1}}}{C_{t_{k-1}}} > \exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right]$, then the previous constraint must be satisfied but it also implies that $m_{t_{k-1}} < 1$ (the upper bound is strictly lower than 1).

- b) *If $\exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] > 1$, then there is no constraint on $m_{t_{k-1}}$.*

The previous proposition shows that, if the multiple is very small, the quantile condition is automatically satisfied (see assertion i). If the value of the multiple is higher (see assertion ii), then, if the market value S decreases (case ii-a), the multiple should be bounded and even smaller than 1 if the previous cushion value is not high. In this latter case, we have to stop the investment in S . If the value of return is sufficiently high (case ii-b), there is no constraint on the multiple.

Proposition 9 *When $0 < C_{t_{k-1}} < L_{t_{k-1}}$, the quantile condition at time t_{k-1} is satisfied according to the following conditions:*

- i) *If $\exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] < 1$, there is no positive solution for $m_{t_{k-1}}$.*
- ii) *If $\exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] > 1$:*
 - If $\frac{L_{t_{k-1}}}{C_{t_{k-1}}} < \exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right]$, then there is no constraint on $m_{t_{k-1}}$.*
 - If $\frac{L_{t_{k-1}}}{C_{t_{k-1}}} > \exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right]$, then $m_{t_{k-1}}$ must satisfy the following constraint:*

$$m_{t_{k-1}} \geq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] - 1}.$$

The previous result corresponds to the search for a minimum portfolio performance. In fact, the threshold $L_{t_{k-1}}$ being positive, we seek permanently to maintain the cushion above a minimum positive value, even if we are more exposed to the risk of fluctuation in the risky asset. Thus, if the initial value of the cushion is below the threshold $L_{t_{k-1}}$ and the return of the risky asset is not sufficiently high, there is no solution. If, on the contrary, the return of the risky asset is relatively high, there is no constraint on the multiple. For an intermediate value of the risky asset return, we must compensate with sufficient exposure. It means that the multiple must be higher than a minimum value.

Remark 10 *As soon as the cushion is positive at time t_{k-1} , the choice of the multiple is very flexible. Therefore, under the quantile condition at time t_{k-1} , other conditions on the multiple can be introduced to better benefit from market variations. As soon as the cushion is negative (small probability), the quantile condition generally does no longer hold, except for small values of $(1 - \epsilon)$. However, in this latter case, this does not correspond to a true insurance condition. Consequently, a practical strategy is to use previous bounds on the multiple when the cushion is positive and to invest the whole portfolio value on the riskless asset, as soon as the cushion is negative.*

4.3 Expected shortfall conditions

We consider now the local Expected Shortfall (ES or CVaR) criterion. Recall that the VaR measure is the potential loss at a given probability level a but VaR does not provide any information about the severity of loss by which it is exceeded. To remedy this issue, we can alternatively introduce the expected shortfall since it is the expectation of losses exceeding the VaR, knowing that the loss is beyond the VaR. In such a framework, first we determine the multiple $m_{t_{k-1}}$ with the VaR condition (see previous propositions):

$$\mathbb{P}[C_{t_k} \leq L_{t_{k-1}} \mid \mathcal{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0] = a, \quad a \in]0, 1[. \quad (19)$$

Second, we minimize the expected shortfall defined by:

$$L_{t_{k-1}} - \mathbb{E}[C_{t_k} \mid \mathcal{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0 \cap C_{t_k} \leq L_{t_{k-1}}]. \quad (20)$$

Such approach is detailed in what follows.

Step 1: We define the set $M_{t_{k-1}}(a, L_{t_{k-1}})$ corresponding to the previous local quantile condition.

Step 2: Then, we calculate the local expected shortfall. Let us consider the case $L_{t_{k-1}} \geq 0$. Recall that:

$$C_{t_k} = C_{t_{k-1}} [1 + m_{t_{k-1}}(R_{t_k} - 1)].$$

Therefore we get:

$$\mathbb{E}[L_{t_{k-1}} - C_{t_k} \mid C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}} \cap C_{t_k} < L_{t_{k-1}}] =$$

$$\frac{\mathbb{E}[(L_{t_{k-1}} - C_{t_k})\mathbb{I}_{C_{t_k} < L_{t_{k-1}}} | C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}}]}{\mathbb{P}[C_{t_k} < L_{t_{k-1}} | C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}}]}, \quad (21)$$

where $\mathbb{I}$ is the Heavyside function that indicates if C_{t_k} is less than $L_{t_{k-1}}$. Denote $\Phi_L(m_{t_{k-1}})$ by:

$$\Phi_L(m_{t_{k-1}}) = \frac{\mathbb{E}[(L_{t_{k-1}} - C_{t_k})\mathbb{I}_{C_{t_k} < L_{t_{k-1}}} | C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}}]}{\mathbb{P}[C_{t_k} < L_{t_{k-1}} | C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}}]}.$$

Determination of $\Phi_L(m_{t_{k-1}})$:

Since $C_{t_k} = C_{t_{k-1}}[1 + m_{t_{k-1}}(R_{t_k} - 1)]$ and $C_{t_{k-1}} > 0$, then we get the following characterization:

$$C_{t_k} < L_{t_{k-1}} \Leftrightarrow [1 + m_{t_{k-1}}(R_{t_k} - 1)] < \frac{L_{t_{k-1}}}{C_{t_{k-1}}},$$

which is equivalent to:

$$R_{t_k} < \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1.$$

and also to

$$\varepsilon_{t_k} < h_{t_k} \left(\frac{1}{\sigma_{t_k}} \ln \left(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 \right) - \frac{\mu_{t_k}}{\sigma_{t_k}} \right). \quad (22)$$

Denote $l_L(m_{t_{k-1}}) = h_{t_k} \left(\frac{1}{\sigma_{t_k}} \ln \left(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 \right) - \frac{\mu_{t_k}}{\sigma_{t_k}} \right)$. Then, the conditional expectation

$$\mathbb{E}[L_{t_{k-1}} - C_{t_k} | \mathcal{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0 \cap C_{t_k} < L_{t_{k-1}}]$$

is equal to

$$L_{t_{k-1}} - \frac{C_{t_{k-1}} \int_{-\infty}^{l_L(m_{t_{k-1}})} [1 + m_{t_{k-1}}(\exp[\sigma_{t_k} h_{t_k}^{-1}(u) + \mu_{t_k}] - 1)] \times f_{\varepsilon_{t_k}}(u) du}{N(l_L(m_{t_{k-1}}))}.$$

Since ε_{t_k} is Gaussian, we have:

$$f_{\varepsilon_{t_k}}(u) = \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{u^2}{2} \right], \quad N(u) = \int_{-\infty}^u f_{\varepsilon_{t_k}}(v) dv.$$

Then, we have:

$$\begin{aligned} \Phi_L(m_{t_{k-1}}) &= L_{t_{k-1}} - \\ C_{t_{k-1}}(1 - m_{t_{k-1}}) &\frac{m_{t_{k-1}} C_{t_{k-1}} \int_{-\infty}^{l_L(m_{t_{k-1}})} (\exp[\mu_{t_k} + \sigma_{t_k} h_{t_k}^{-1}(u)]) \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du}{N(l_L(m_{t_{k-1}}))}. \end{aligned} \quad (23)$$

Therefore:

$$\Phi(m_{t_{k-1}}) = L_{t_{k-1}} - C_{t_{k-1}}[(1 - m_{t_{k-1}}) + \frac{m_{t_{k-1}}}{N(l_L(m_{t_{k-1}}))} \times I_L(m_{t_{k-1}})],$$

with

$$I_L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{l_L(m_{t_{k-1}})} \exp[\mu_{t_k} + \sigma_{t_k} h_{t_k}^{-1}(u)] \exp(\frac{-u^2}{2}) du.$$

Note that, for $L_{t_{k-1}} = 0$, we get:

$$l_0(m_{t_{k-1}}) = h_{t_k} \left(\frac{1}{\sigma_{t_k}} \ln \left(\frac{-1}{m_{t_{k-1}}} + 1 \right) - \frac{\mu_{t_k}}{\sigma_{t_k}} \right).$$

where $l_0(m_{t_{k-1}})$ does not depend on the cushion value $C_{t_{k-1}}$ at time t_{k-1} , and we get:

$$\Phi_0(m_{t_{k-1}}) = -C_{t_{k-1}}[(1 - m_{t_{k-1}}) + \frac{m_{t_{k-1}}}{N(l_0(m_{t_{k-1}}))} \times I_0],$$

with

$$I_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{l_0(m_{t_{k-1}})} \exp[\mu_{t_k} + \sigma_{t_k} h_{t_k}^{-1}(u)] \exp(\frac{-u^2}{2}) du.$$

Finally, we minimize the local expected shortfall for conditional multiples satisfying the local quantile condition (i.e. $m_{t_{k-1}} \in M_{t_{k-1}}(a, L_{t_{k-1}})$).

$$\widetilde{\mathbb{E}}_{\alpha}[C_{t_k}] =$$

$$\text{Min}_{\{m_{t_{k-1}} \in M_{t_{k-1}}(a, L_{t_{k-1}})\}} (L_{t_{k-1}} - \mathbb{E}[C_{t_k} | \mathcal{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0 \cap C_{t_k} \leq L_{t_{k-1}}]).$$

The expected shortfall is minimized for values of the multiple variable $m_{t_{k-1}}^*$ which lie between the two bounds of the multiple variable $m_{t_{k-1}}$ and satisfy the VaR condition.

5 Empirical illustrations

5.1 Estimation of hedge funds returns by Johnson system

We determine the family and estimate the parameters of Johnson distributions for the 10 sub-indices of hedge funds with three methods, namely the method of moments using the algorithm of Hill *et al.* (1976), the method of quantile of Wheeler (1980) and the other method of quantile considered by Slifker and Shapiro (1980). Since we have three different estimation methods, we perform a fit test of χ^2 to differentiate them.

Our study is conducted on the Credit Suisse Hedge Fund Index provided by Credit Suisse. These indexes provide the monthly returns of the ten main families of alternative strategies identified by this data provider. These indices are Convertible Arbitrage (CA), Dedicated Short Bias (DSB), Emerging Markets (EM), Equity Market Neutral (EMN), Event Driven (ED), Fixed Income Arbitrage (FIA), Global Macro (GM), Long / Short Equity (LSE), Managed Futures (MF) and Multi-Strategy (MF).

The global index constructed from the monostrategy indices is used to represent all hedge fund strategies. These indices are constructed using the TASS database which includes more than 3000 U.S. and offshore hedge funds. The characteristic of these indices is that they are known by the transparency guaranteed by Credit Swiss in their communication and construction mode.

These indices include only funds having at least \$ 10 million under management and providing audited financial reports. After this filtering, 400 funds on average are retained. The procedure for selecting funds. is renewed quarterly to update calculations. In fact, if a fund no longer satisfies expectations of financial reporting, it is systematically excluded. As to the classification of funds by major styles of strategy, it is realized on the basis of declarations of the managers. Funds are then weighted by their capitalization and presented net of commissions.

The study of indices is conducted on the period from December 1994 until December 2013, which gives us the possibility to work on 240 monthly observations. This period covers a very significant number of financial crises.

Table 1: Distribution type and Johnson parameters estimated by the method of moments

	γ	δ	ξ	λ	Type
Convertible Arbitrage	0.8394	1.2845	0.0178	0.0231	SU
Dedicated Short Bias	-1.5101	2.5148	-0.0674	0.0925	SU
Event Driven	1.2321	1.4236	0.0288	0.0168	SU
Emerging Markets	0.4364	1.4486	0.0283	0.0554	SU
Equity Market Neutral	0.4745	1.4760	0.0322	0.0617	SU
Fixed Income Arbitrage	1.8461	1.3189	0.0259	0.0084	SU
Global Macro	-0.0221	1.4753	0.0087	0.0312	SU
Long/Short Equity	-0.0259	1.6319	0.0071	0.0443	SU
Managed Futures	0	1	0.0333	0.0047	SN
Multi-Strategy	0.9981	1.7174	0.0235	0.0231	SU

Table 2: Distribution type and Johnson parameters estimated by the method of Wheeler (1980)

	γ	δ	ξ	λ	Type
Convertible Arbitrage	0.2850	0.9563	0.0128	0.0116	SU
Dedicated Short Bias	0	4.8091	-0.2286	0.2197	SL
Event Driven	0.6080	1.3611	0.0202	0.0201	SU
Emerging Markets	0.4059	1.1549	0.0273	0.0382	SU
Equity Market Neutral	0.3987	1.1663	0.0291	0.0439	SU
Fixed Income Arbitrage	0.3502	1.1513	0.0099	0.0100	SU
Global Macro	-0.0087	1.1211	0.0093	0.0188	SU
Long/Short Equity	-0.0908	1.5549	0.0052	0.0402	SU
Managed Futures	0	1	0.0041	0.0325	SN
Multi-Strategy	0.3350	1.0656	0.0133	0.0125	SU

Table 3: Distribution type and Johnson parameters estimated by the method of Slifker and Shapiro (1980)

	γ	δ	ξ	λ	Type
Convertible Arbitrage	0.2558	0.9207	0.0125	0.0116	SU
Dedicated Short Bias	-1.9022	1.4407	-0.1321	0.0683	SL
Event Driven	0.7837	1.5433	0.0220	0.0219	SU
Emerging Markets	0.2825	1.3488	0.0213	0.0468	SU
Equity Market Neutral	0.2574	1.3361	0.0210	0.0496	SU
Fixed Income Arbitrage	0.2444	0.9018	0.0090	0.0074	SU
Global Macro	-0.0045	0.9383	0.0096	0.0140	SU
Long/Short Equity	-0.0434	1.5932	0.0070	0.0407	SU
Managed Futures	-0.0141	2.4468	0.0045	0.1308	SU
Multi-Strategy	0.3745	1.2326	0.0139	0.0157	SU

Tables (1), (2) and (3) show that the distribution type and parameter values may vary depending on the estimation method that is used. Indeed, for Dedicated Short Bias Index, according to the method of moments we find that the distribution is log-normal (*SL*) while with the method of Wheeler (1980) and the method of Slifker and Shapiro (1980), we find that the distribution is unbounded (*SU*). We also find that most of hedge fund returns have *SU* distribution. In order to choose the best method that corresponds to each index, we use the χ^2 adequacy test. With this test, we retain the method that minimizes the value of χ^2 .

Table 4: Values of χ^2 test according to different estimation methods

	Moments	Wheeler	Slifker & Shapiro
Convertible Arbitrage	15.7779	1.5259	1.2867
Dedicated Short Bias	7.1041	16.2136	15.9684
Event Driven	13.1077	6.3583	7.9988
Emerging Markets	6.3238	10.2190	11.7220
Equity Market Neutral	5.8309	10.0031	11.8772
Fixed Income Arbitrage	59.0119	2.3502	2.5957
Global Macro	16.0726	4.7018	2.1119
Long/Short Equity	4.0751	2.4113	2.5970
Managed Futures	3.3392	3.3392	3.0278
Multi-Strategy	6.8135	2.6522	0.5765

Table (4) shows that to each index corresponds a particular estimation method.

In order to ensure the effectiveness of the Johnson transformation, we transform the series of returns of Multi-Strategy index to a series of random variables the normality of which we test (see Equations 1 and 2).

Figure 1: Histogram and QQ plot of transformed Multi Strategy index

Table 5: Statistical characteristics of the series of variables obtained by Johnson transformation

Four first moments	
Mean	-0.0206
Standard deviation	0.9991
Skewness	0.0037
kurtosis	2.8759
Kolmogorov-Smirnov test	
Normality	yes
p-value	0.6987
Jarque-Bera test	
Normality	yes
p-value	0,5000

Figure (1) and Table (5) confirm the normality of distribution of random variables obtained by the Johnson transformation. This is in favor of the proposed approach based on Johnson distributions.

5.2 CPPI portfolio payoffs under quantile criterion

To study the CPPI method with conditional multiple within Johnson distributions and temporal dependence of financial series, we choose the Multi Strategy index as representative risky asset. We introduce the $EGARCH(1,1)$ process with Johnson type innovation to model the conditional volatility of log-returns. In what follows, we illustrate empirically our theoretical results. We simulate 1000 paths of CPPI portfolio values using conditional quantile criterion. The initial values of the portfolio and the price of the risky asset are equal to 100. The period of the investment corresponds to 5 years and the guarantee rate required at maturity is 100% of the initial capital. We plot both the cdf of portfolio returns and the cdf of the ratio of portfolio values for a threshold corresponding to $L_{t_{k-1}} = 0.75C_{t_{k-1}}$.

Figure 2: Distribution functions of final payments and ratios of CPPI portfolios with variable multiple under quantile criterion and different levels of loss probability

Figure (2) shows that, for a given threshold L , the portfolio with an acceptable loss probability of 10% stochastically dominates (approximately) at the first order the portfolio with an acceptable loss probability of 5%. Indeed, portfolio associated to 5% can dominate the portfolio associated to 10% only for significant bearish market. This can be explained by the fact that the more risk averse the investor, the higher the proportion of his investment he wants to guarantee. This result is mainly due to the multiple values used in portfolio management. Indeed, for portfolios of 5%, multiples are lower than those of 10%. This leads to lower exposure to risk. We also note that for both levels of probability, the value of the initial investment is guaranteed in both cases. This is due to the method of choice of multiple.

Figure 3: Distribution functions of CPPI portfolio payoffs with conditionnal multiple under quantile criterion and different levels of threshold

Figure (3) illustrates that, for a given probability level, the final value of the portfolio depends on the threshold L . The portfolio with a threshold $L_{t_{k-1}} = 65\% C_{t_{k-1}}$ dominates the two portfolios with thresholds 75% of $C_{t_{k-1}}$ and 85% of $C_{t_{k-1}}$ for bullish market. On the contrary, it is dominated by the two portfolios for bearish market. The portfolio with threshold $L_{t_{k-1}} = 75\% C_{t_{k-1}}$ dominates the portfolio of threshold 65% of $C_{t_{k-1}}$ if the market fluctuation is moderate. This shows that an increase in the threshold level allows to increase probability to recover the initial investment but reduces it for bullish market. This seems logical because, to keep past performance, we must be exposed to less risk. This falls well within the framework of portfolio insurance. We also note that all portfolios were able to recover the initial investment. This illustrates the robustness of the method with CPPI multiple variable that takes account both of the conditional volatility of the risky asset and also the level of risk aversion of the investor.

5.3 Payoff of the CPPI portfolio under expected shortfall criterion

In the previous section, we have proved that the multiple chosen according to the expected shortfall criterion can be obtained by minimizing the function $\Phi_L(m_{t_{k-1}})$. This function is the conditional expectation of the difference between the threshold value $L_{t_{k-1}}$ and the cushion C_{t_k} . In what follows, we present the different values of multiple under local expected shortfall criterion.

Recall that, under this criterion, the minimization problem corresponds to:

$$\mathbb{E}[L_{t_{k-1}} - C_{t_k} | C_{t_{k-1}} > 0 \cap \mathcal{F}_{t_{k-1}} \cap C_{t_k} < L_{t_{k-1}}].$$

and the function to minimize is given by:

$$\Phi_L(m_{t_{k-1}}) = L_{t_{k-1}} - C_{t_{k-1}}[(1 - m_{t_{k-1}}) + \frac{m_{t_{k-1}}}{N(l_L(m_{t_{k-1}}))} \times I_L(m_{t_{k-1}})],$$

We still choose the threshold value $L_{t_{k-1}}$ as a percentage P of cushion $C_{t_{k-1}}$. This implies that we want to keep a percentage of the final realized performance. We study the three possible cases for P : $P = 0$ (in this case $L_{t_{k-1}} = 0$), $P \leq 1$ and $P > 1$. To determine the multiple that minimizes the function $\Phi_L(m_{t_{k-1}})$, we set an interval $[1, 10]$ in which can vary possible values of multiple. We find that, if $P \leq 1$, the multiple often takes the value of the upper bound of the interval and even by changing the boundaries of this interval, the multiple always takes the value of the upper bound. Furthermore, by putting $P > 1$, we obtain different values of multiple. To determine the effect of $L_{t_{k-1}}$ on the choice of multiple, we choose three different values of P are: 110%, 115% and 120%. We simulate 1000 paths of CPPI portfolio values with a variable multiple selected according to expected shortfall criterion. The initial values of the portfolio and the price of the risky asset are equal to 100.

Figure 4: Cdf of CPPI portfolios with variable multiple under expected shortfall criterion with different values of L

Figure (4) shows that the portfolio with $L_{t_{k-1}} = 120\% C_{t_{k-1}}$ dominates portfolios with $L_{t_{k-1}} = 110\% C_{t_{k-1}}$ and $L_{t_{k-1}} = 115\% C_{t_{k-1}}$ when the market rises. However, it is dominated by these two portfolios for bearish market. We explain these results by the fact that if we want to get better performance than the last realized performance, we must increase our risk exposure by using higher values of multiple. We also find that, in all cases, the probability of losing the initial investment is equal to zero. Even if we are much exposed to risk, at maturity we can reimburse the full amount initially invested. This also proves the robustness of the method we use to determine the conditional multiple.

5.4 Comparison between final values of the CPPI according to the method of choice of multiple

To evaluate the different methods of multiple choice, we make a comparison between the three methods that we have presented: fixed multiple, multiple with quantile criterion and multiple with expected shortfall criterion.

Figure 5: Cdf of CPPI portfolio payoffs under three methods to choose the multiple

Figure (5) shows that the portfolio with expected shortfall criterion dominates the portfolio with quantile criterion and the portfolio with fixed multiple. In fact, we have a probability equal to 90% that the final portfolio value with expected shortfall criterion is higher than the portfolio with quantile criterion and a probability equal to 75% that it is higher than the portfolio with fixed multiple. This shows that, even if the investor is exposed to more risk, he can achieve a good performance while ensuring a better hedge against downside risk by choosing a multiple variable that minimizes the expectation of the difference between the value of threshold and that of the cushion. We also find that the portfolio with quantile criterion dominates portfolio with fixed multiple when the market is bearish. We have a probability equal to 80% that the final value of the quantile portfolio is higher than the portfolio with a fixed multiple. When the market increases moderately, the portfolio associated to a fixed multiple dominates the portfolio corresponding to quantile criterion. From this comparison, we can conclude that the method based on a variable multiple can provide better performance and hedging against the risk, whatever the evolution of the risky asset. This is explained by the introduction of the conditional volatility estimated by $EGARCH(1,1)$ with a distribution of innovation that takes the higher order moments into account.

6 Conclusion

We have examined CPPI strategy on hedge funds. Due to the non normality of their returns, we have introduced Johnson distributions. We have also assumed that rebalancing times are discrete, which corresponds to actual practice. We have provided upper bounds on conditional multiple, by introducing gap risk controlled by means of both quantile and expected shortfall criteria. The upper bound of the multiple is a function of the given probability level, of the threshold L and of the conditional volatility of logarithmic returns of risky asset. This latter one is estimated from the $EGARCH(1,1)$ with innovations that follow Johnson distribution SU . Results show that by taking these criteria into account, we can provide a better portfolio insurance while having a better ability to profit from market increases.

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Appendix

Proofs of propositions 8 and 9

In what follows, the cushion at time t_{k-1} is strictly positive ($C_{t_{k-1}} > 0$).

Case 1: $C_{t_{k-1}} > L_{t_{k-1}}$.

The local quantile condition is the following:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} \right] \geq (1 - a),$$

which is equivalent to:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}} (R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \right] \geq (1 - a).$$

At time t_k , we have two cases:

$$\begin{cases} R_{t_k} < 1 : S_{t_k} < S_{t_{k-1}} \text{ (} S \text{ decreases)}, \\ R_{t_k} > 1 : S_{t_k} > S_{t_{k-1}} \text{ (} S \text{ increases)}. \end{cases}$$

We assume that the multiple $m_{t_{k-1}}$ is higher than 1, (thus, it is non negative).

Then, we deduce:

$$\begin{aligned} & \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}} (R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \right] \\ &= \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}} (R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (R_{t_k} - 1) > 0 \right] + \\ & \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}} (R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (R_{t_k} - 1) < 0 \right]. \end{aligned}$$

- 1.1. If $C_{t_{k-1}} > L_{t_{k-1}}$, then:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}} (R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \right] =$$

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}} [R_{t_k} - 1 > 0] + \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}} (R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (R_{t_k} - 1) < 0 \right].$$

Note that we have:

$$m_{t_{k-1}} (R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \Leftrightarrow R_{t_k} > \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1.$$

Then, if the condition $\frac{\frac{L_{t_{k-1}}-1}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 > 0$ is fulfilled and we have: $R_{t_k} > \frac{\frac{L_{t_{k-1}}-1}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1$.⁵

1.1.1. If we have $\frac{\frac{L_{t_{k-1}}-1}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 < 0$, then the condition $R_{t_k} > \frac{\frac{L_{t_{k-1}}-1}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1$ is satisfied and:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} \right] = 1 > (1-a).$$

1.1.2. Examine the case $C_{t_{k-1}} > 0$, $C_{t_{k-1}} > L_{t_{k-1}}$ and $\frac{\frac{L_{t_{k-1}}-1}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 > 0$. We have:

$$\begin{aligned} & \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}} (R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \right] \\ = & \mathbb{P}^{\mathcal{F}_{t_{k-1}}} [R_{t_k} - 1 > 0] + \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[1 > R_{t_k} > \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 \right] \\ = & \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[R_{t_k} > \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 \right], \end{aligned}$$

which is equivalent to:

$$\begin{aligned} & \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[\exp \left[h_{t_k}^{-1} (\varepsilon_{t_k}) \sigma_{t_k} + \mu_{t_k} \right] > \frac{\frac{L_{t_{k-1}}-1}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 \right] \geq (1-a) \\ = & \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[\varepsilon_{t_k} > h_{t_k} \left(\frac{1}{\sigma_{t_k}} \ln \left(\frac{\frac{L_{t_{k-1}}-1}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 \right) - \frac{\mu_{t_k}}{\sigma_{t_k}} \right) \right] \geq (1-a) \\ = & 1 - \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[\varepsilon_{t_k} < h_{t_k} \left(\frac{1}{\sigma_{t_k}} \ln \left(\frac{\frac{L_{t_{k-1}}-1}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 \right) - \frac{\mu_{t_k}}{\sigma_{t_k}} \right) \right] \geq (1-a) \\ = & \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[\varepsilon_{t_k} < h_{t_k} \left(\frac{1}{\sigma_{t_k}} \ln \left(\frac{\frac{L_{t_{k-1}}-1}{C_{t_{k-1}}}-1}{m_{t_{k-1}}} + 1 \right) - \frac{\mu_{t_k}}{\sigma_{t_k}} \right) \right] \leq 1 - (1-a). \end{aligned}$$

⁵Note that this condition is satisfied as soon as $L_{t_{k-1}} > 0$.

Denote by N the cdf of the standard Gaussian distribution (the cdf of ε_{t_k}). We have:

$$N \left[h_{t_k} \left(\frac{1}{\sigma_{t_k}} \ln \left(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 \right) - \frac{\mu_{t_k}}{\sigma_{t_k}} \right) \right] \leq 1 - (1 - a),$$

which is equivalent to:

$$h_{t_k} \left(\frac{1}{\sigma_{t_k}} \ln \left(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 \right) - \frac{\mu_{t_k}}{\sigma_{t_k}} \right) \leq (N)^{-1} (1 - (1 - a)).$$

and also equivalent to:

$$\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} \leq \exp \left[\sigma_{t_k} h_{t_k}^{-1} (N^{-1} (1 - (1 - a))) + \mu_{t_k} \right] - 1.$$

Recall that $q_{(\epsilon, T)} = N^{-1} \left[(1 - (1 - a))^{\frac{1}{T}} \right]$.

Finally, we have:

1-1) If $\exp \left[h_{t_k}^{-1} \left(q_{(a, T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] < 1$,

If $\frac{L_{t_{k-1}}}{C_{t_{k-1}}} < \exp \left[h_{t_k}^{-1} \left(q_{(a, T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right]$, then $m_{t_{k-1}}$ must satisfy the following constraint:

$$m_{t_{k-1}} \leq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp \left[h_{t_k}^{-1} \left(q_{(a, T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] - 1}.$$

If $\frac{L}{C_{t_{k-1}}} > \exp \left[h_{t_k}^{-1} \left(q_{(a, T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right]$, then the previous constraint must be satisfied but also involves more that $m_{t_{k-1}} < 1$.

1-2) If $\exp \left[h_{t_k}^{-1} \left(q_{(a, T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] > 1$, then there is no constraint for $m_{t_{k-1}}$.

Case 2: $0 < C_{t_{k-1}} < L_{t_{k-1}}$, Therefore: $L_{t_{k-1}} > 0$.

As previously, the quantile condition is given by:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} \right] \geq (1 - a),$$

which is equivalent to:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}} (R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \right] \geq (1 - a)$$

At time t_k , we have two cases:

$$\begin{cases} R_{t_k} < 1 : S_{t_k} < S_{t_{k-1}} \text{ (S decreases)}, \\ R_{t_k} > 1 : S_{t_k} > S_{t_{k-1}} \text{ (S increases)}. \end{cases}$$

We assume that the multiple $m_{t_{k-1}}$ is non negative.

$$\begin{aligned}
& \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}}(R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \right] \\
= & \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}}(R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (R_{t_k} - 1) > 0 \right] + \\
& \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}}(R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (R_{t_k} - 1) < 0 \right].
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
& \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}}(R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \right] \\
= & \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}}(R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \cap (R_{t_k} - 1) > 0 \right] \\
= & \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[m_{t_{k-1}}(R_{t_k} - 1) > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \right].
\end{aligned}$$

Consequently, we get the equivalence between the two following conditions:

$$\begin{aligned}
& \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} > \frac{L_{t_{k-1}}}{C_{t_{k-1}}} \right] \geq (1 - a), \\
& \mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[R_{t_k} \geq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 \right] \geq (1 - a).
\end{aligned}$$

This is equivalent to:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[R_{t_k} \geq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 \right] \geq (1 - a).$$

Thus, the quantile condition is equivalent to:

$$\mathbb{P}^{\mathcal{F}_{t_{k-1}}} \left[\varepsilon_{t_k} > h_{t_k} \left(\frac{1}{\sigma_{t_k}} \ln \left(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 \right) - \frac{\mu_{t_k}}{\sigma_{t_k}} \right) \right] \geq (1 - a)$$

And also to:

$$h_{t_k} \left(\frac{1}{\sigma_{t_k}} \ln \left(\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} + 1 \right) - \frac{\mu_{t_k}}{\sigma_{t_k}} \right) \leq N^{-1} (1 - (1 - a)).$$

We obtain:

$$\frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{m_{t_{k-1}}} \leq \exp \left[h_{t_k}^{-1} \left(q_{(\alpha, T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] - 1.$$

Finally, we have:

2-1) If $\exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] < 1$, there is no positive solution for $m_{t_{k-1}}$

2-2) If $\exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] > 1$, then :

If $\frac{L_{t_{k-1}}}{C_{t_{k-1}}} < \exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right]$, then there is no constraint for $m_{t_{k-1}}$

If $\frac{L_{t_{k-1}}}{C_{t_{k-1}}} > \exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right]$, then $m_{t_{k-1}}$ must satisfy the following constraint:

$$m_{t_{k-1}} \geq \frac{\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1}{\exp \left[h_{t_k}^{-1} \left(q_{(a,T)}^\varepsilon \right) \sigma_{t_k} + \mu_{t_k} \right] - 1}.$$