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OPTIMAL RISK SHARING WITH OPTIMISTIC AND PESSIMISTIC DECISION MAKERS

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ABSTRACT

We prove that under mild conditions individually rational Pareto optima will exist even in presence of non-convex preferences. We consider decision makers dealing with a countable flow of payoffs or choosing among financial assets whose outcomes depend on the realization of a countable set of states of the world. Our conditions for the existence of Pareto optima can be interpreted as a requirement of impatience in the first context and of some pessimism or not unrealistic optimism in the second context. A non-existence example is provided when, in the second context, some decision maker is too optimistic. We furthermore show that at an individually rational Pareto optimum at most one strictly optimistic decision maker will avoid financial ruin. Considering a risky context this entails that even if risk averters will share risk in a comonotonic way as usual, at most one classical strong risk lover will avoid ruin.

1 Introduction

Our aim is to study how the interaction between optimism and pessimism (and also risk propensity and risk aversion) could affect the allocation of economic resources. In order to do this, we will analyse which are the properties of the socially desired allocations, in the sense of being Pareto optimal (PO) for which any decision maker (DM) of the economy is not worse than with the initial endowments of goods. One should remark that the presence of non-convex preferences can cause technical issues to ensure the existence of general equilibrium for the respective economy, but even in these cases, it is possible to ask about the existence of individually rational Pareto optimal (IRPO) - which is a lesser demanding concept than equilibrium, as it does not require the existence of a price system - and on the properties that should be verified by such allocations.

We will model a pure exchange economy as did Bewley (1972), with a finite number of DMs and an infinite dimensional consumption space. More specifically, the consumption space is the set of non-negative real bounded sequences ℓ_+^∞ , and, thus, we can encompass the analysis of a decision problem with infinitely many states or with an infinite flow of payments along the time.

This paper has two main results. The first is about the existence of IRPO allocations for economies without the usual requirements as completeness or convexity over decision makers' preferences. A central hypothesis for this result is that preferences display upper impatience. Indeed, Araujo (1985) provides examples of economies for which there is no IRPO due to the lack of impatience¹. Under the presence of ambiguity (as modelled by Schmeidler (1986)), it is well known that, if the associated capacity is concave, the utility function that represents the DM's preferences can be written as a maximum, on the set of DM's priors², of the respective expected utilities. In this case, the sufficient hypothesis in order to ensure upper impatience is outer continuity of the respective capacity at $\emptyset$. Additionally, we will illustrate the need of this condition with an example in that the

¹It is important to remark that, in this work, as in Brown and Lewis (1981), the term impatience is simply a specialization of the broader concept of myopia. Some authors such as Chateauneuf and Rébillé (2004) propose alternative definitions for these two concepts.

²That is, the capacity core.

presence of a too optimistic agent will lead to the non-existence of IRPO.

We then consider the presence of reasonably but strictly optimistic DM. Our second main result, Theorem 2, shows that at IRPO allocations at most such strictly optimistic DM will avoid ruin.

Then specializing in the context of risk, we obtain that as in the standard case with finite states and without non-convexities studied by Chateauneuf, Dana and Tallon (2000), at an IRPO, the allocations of the risk averters are comonotonic. The main feature of risk-sharing between risk averters and risk lovers as settled in Theorem 3, is that for classical strong risk lovers, i.e. those with strictly concave preferences, again ruin could be avoided by at most one of them.

The paper is structured as follows: In Section 2 we present sufficient conditions in order to guarantee that there are IRPO and provide the proof of existence. In Section 3 we relate the condition of weak* sequentially upper semi continuity of preferences and the notions of myopia and impatience. Section 4 gives some examples of weak* sequentially upper-semicontinuous preferences. Finally in Section 5 we analyse the optimal allocations of strictly optimistic DM and then specialize to the context of risk.

2 Existence of IRPO allocations under weak* sequentially upper-semicontinuous preferences

REMARK 1: One will find in Appendix A some brief recalls concerning ℓ^∞ the Banach space of real bounded sequences, and also the weak* topology and the Mackey topology, which are respectively the coarsest and the finest topology on ℓ^∞ for which the dual is ℓ^1 the Banach space of absolutely convergent real sequences.

REMARK 2: Some readers might wonder why we do not attempt to prove existence of IRPO allocations for the greatest class of Mackey sequentially upper-semicontinuous preferences.

Actually it seems to be a “Folk Theorem” that on ℓ^∞ , the weak* $\sigma(\ell^\infty, \ell^1)$ sequential convergence and the Mackey $\tau(\ell^\infty, \ell^1)$ sequential convergence coincide. We propose in Appendix B a proof of this result, which justifies the use of the more tractable weak* $\sigma(\ell^\infty, \ell^1)$ sequential convergence.

Assumptions

- (1) For every $i = 1, \dots, m$, $\succeq_i$ is a transitive reflexive preorder on ℓ_+^∞ .
- (2) For every i , every $y_i \in \ell_+^\infty$, $\{x_i \in \ell_+^\infty \mid x_i \succeq_i y_i\}$ is weak* sequentially closed.

Let $w_i \in \ell_+^\infty$ be the initial endowments of individual $i = 1, \dots, m$.

Let $\mathcal{A} = \{(x_1, \dots, x_i, \dots, x_m) \in (\ell_+^\infty)^m : \sum_{i=1}^m x_i = w \text{ where } w := \sum_{i=1}^m w_i\}$ be the set of feasible allocations.

A feasible allocation $(x_1, \dots, x_i, \dots, x_m)$ is said to be:

1. *Pareto optimal* (PO) if there is no other allocation $(z_1, \dots, z_m)$ satisfying $z_i \succsim_i x_i \forall i$ and $z_k \succ_k x_k$ for some k .
2. *Individually rational Pareto optimal* (IRPO) if it is PO and $x_i \succsim_i w_i \forall i$.

Now, we can state our first main result:

THEOREM 1: *Under Assumptions (1) and (2), IRPO allocations exist.*

The proof will be performed by a similar method as the one used pages 220 - 221 by Aliprantis and Burkinshaw (1978) for weakly u.s.c. preferences.

The proof results from two lemmas:

LEMMA 1: *The set $\mathcal{A}$ is compact in $(\ell^\infty)^m$ endowed with its product topology $\mathcal{P}$, where each ℓ^∞ is endowed with the weak* topology.*

PROOF: Let

$$\pi = \sup_{n \in \mathbb{N}} w(n),$$

note that the Edgeworth box $[0, w] = \{x \in \ell^\infty : 0 \leq x \leq w\}$ is included in the closed ball $\bar{B}(0, \pi)$. Since $\bar{B}(0, \pi)$ is weak* compact, it will follow that $[0, w]$ is w^* -compact as soon

as we prove that $[0, w]$ is w^* -closed. Let (x^α) be a net in $[0, w]$ w^* -converging towards x , therefore $x^\alpha(n)$ converges towards $x(n)$ for every n in $\mathbb{N}$, and thus $0 \leq x(n) \leq w(n)$, so $x \in [0, w]$, and actually $[0, w]$ is w^* closed, hence w^* compact. Note that $\mathcal{A} \subset [0, w]^n$, so $\mathcal{A}$ is $\mathcal{P}$ - compact if $\mathcal{A}$ is $\mathcal{P}$ - closed.

Let $\{(x_i^\alpha), i = 1, \dots, m\}$ a net in $\mathcal{A}$ converging towards $(x_i)_{i=1, \dots, m}$ for the product topology $\mathcal{P}$, hence for each $i = 1, \dots, m, x_i^\alpha \xrightarrow{w^*} x_i$ so as seen above $x_i \in \ell_+^\infty$, since $x_i^\alpha \in [0, w] \forall \alpha$ implies $x_i \in [0, w]$.

Furthermore since $x_i^\alpha \xrightarrow{w^*} x_i \in \ell_+^\infty \forall i$, and

$$\sum_{i=1}^m x_i^\alpha = w \forall \alpha$$

it turns out since (ℓ^∞, w^*) is a topological vector space, that indeed

$$\sum_{i=1}^m x_i = w,$$

so $\mathcal{A}$ is $\mathcal{P}$ - closed, and finally $\mathcal{A}$ is $\mathcal{P}$ - compact. ■

The set of all individually rational allocations is denoted $\mathcal{A}_r$, i.e. , $\mathcal{A}_r = \{(x_1, \dots, x_m) \in \mathcal{A} : x_i \succsim_i w_i \text{ for each } i\}$. One can observe that $\mathcal{A}_r$ is non-empty.

LEMMA 2: *The set $\mathcal{A}_r$ is $\mathcal{P}$ - compact and individually rational Pareto efficient allocations exist.*

PROOF: Note that $\mathcal{A}_r = \mathcal{A} \cap \prod_{i=1}^m \{x_i \in \ell_+^\infty, x_i \succsim w_i\} \cap \bar{B}(0, \pi)$.

Since the weak* topology on $\bar{B}(0, \pi)$ (see e.g. J.B. Conway “A Course in Functional Analysis” Exercise 4, p.136 on else the proof of 6.34 Theorem, p.254, in Aliprantis and Border “Infinite Dimensional Analysis”) is metrizable, it turns out that $\{x_i \in \ell_+^\infty, x_i \succsim w_i\} \cap \bar{B}(0, \pi)$ is weak* compact since $\{x_i \in \ell_+^\infty, x_i \succsim w_i\}$ is weak* sequentially closed. Therefore $\prod_{i=1}^m \{x_i \in \ell_+^\infty, x_i \succsim w_i\} \cap \bar{B}(0, \pi)$ is $\mathcal{P}$ - closed, hence $\mathcal{A}_r$ is $\mathcal{P}$ - closed in $\mathcal{A}$ $\mathcal{P}$ - compact, hence $\mathcal{A}_r$ is compact.

Let us prove now that individually rational Pareto efficient allocations exist.

Let us define a partial preorder $\succsim$ on $\mathcal{A}_r$ by:

$$(x_1, \dots, x_i, \dots, x_m) \succsim (y_1, \dots, y_i, \dots, y_m) \text{ if } x_i \succsim y_i, \forall i = 1, \dots, m.$$

So an individually rational Pareto efficient allocation exists if and only if $\mathcal{A}_r$ has a maximal element.

From Zorn's lemma, we know that the preordered set $(\mathcal{A}_r, \succsim)$ will have a maximal element if every totally preordered subset of $(\mathcal{A}_r, \succsim)$ is majorized.

Let (ζ_α) be a totally preordered subset of $(\mathcal{A}_r, \succsim)$ and define for each α $C_\alpha = \{t \in \mathcal{A}_r, t \succsim \zeta_\alpha\}$; from weak* sequential upper-semicontinuity of each preference i , it turns out as above that C_α is a $\mathcal{P}$ - closed subset of the $\mathcal{P}$ -compact set $\mathcal{A}_r$. Since (ζ_α) is totally preordered, it comes that the intersection of a finite number of C_α is non-empty. Consequently, $C = \bigcap_{\alpha} C_\alpha$ is non-empty, and any $t \in C$ majorizes (ζ_α) , so a maximal element does exist and consequently there exist IRPO allocations. ■

3 Existence of individually rational Pareto efficient allocations under upper semi-myopic preferences

Let us now recall the central notion introduced by Brown and Lewis (1981) (see also Araujo (1985)) under the denomination of strong myopia (or in a sequential context, strong impatience) and quoted also in Araujo, Novinski and Páscoa (2011) under the denomination of upper semi-myopia.

DEFINITION: A preference relation $\succeq$ on ℓ_+^∞ , is upper semi-myopic if $\forall (x, y) \in \ell_+^\infty \times \ell_+^\infty$, $x \succ y$, $z \in \ell_+^\infty$, implies that there exists n large enough such that $x \succ y + z_{E_n}$ where $z_{E_n}(m) = 0$ if $m \leq n$ and $z(m)$ if $m > n$.

Note that in case of preferences over a countable flow of payoffs, the previous definition clearly fits to the notion of upper semi-myopia, while in case of preferences over financial assets whose outcomes depend on the realization of a countable set of states of the world the previous definition would model neglecting gains on “small events” i.e. some pessimism or not unreasonable optimism.

Throughout the paper we will assume now that preferences satisfy Assumption A1:

ASSUMPTION A1: Preferences $\succsim$ on ℓ_+^∞ are complete, transitive, monotone (i.e. $x, y \in$

ℓ_+^∞ , $x \geq y \Rightarrow x \succsim y$) and norm continuous.

For sake of completeness, we state and prove a key property:

PROPOSITION 1: *For preferences $\succsim$ on ℓ_+^∞ satisfying A1 the following assertions are equivalent:*

- (1) $\succsim$ is weak* sequentially upper-semicontinuous
- (2) $\succsim$ is upper semi-myopic

PROOF:

(1) $\Rightarrow$ (2)

Let $x, y \in \ell_+^\infty$ with $x \succ y$ and let $z \in \ell_+^\infty$.

Assume that for any n , $y + z_{E_n} \succsim x$, clearly $z_{E_n} \xrightarrow{w*} 0$, then from (1) $y + 0 \succsim x$ i.e. $y \succsim x$ a contradiction, which completes the proof. $\blacksquare$

(2) $\Rightarrow$ (1)

Assume that there exist $x \in \ell_+^\infty$ and a sequence (y^m) , $y^m \in \ell_+^\infty$, $y^m \xrightarrow{w*} y$, $y \in \ell_+^\infty$ such that $y^m \succsim x \ \forall m$ and $x \succ y$. Since $y^m \xrightarrow{w*} y$, the sequence is bounded let us say by $k \in \mathbb{R}_+$, therefore for any n , $y^m \leq y_{E_n^c}^m + k \mathbf{1}_{E_n}$ and by monotonicity of $\succsim$, one gets $y^m \precsim y_{E_n^c}^m + k \mathbf{1}_{E_n}$. Hence $x \precsim y_{E_n^c}^m + k \mathbf{1}_{E_n} \ \forall m$ for any fixed n .

Let $y = (y_1, \dots, y_n, y_{n+1}, \dots)$, since $y^m(i) \rightarrow y_i \ \forall i \in \mathbb{N}$, norm continuity of $\succsim$ therefore implies:

$$x \precsim (y_1, \dots, y_n, k, \dots, k, \dots) \ \forall n \in \mathbb{N}.$$

But letting $z = (k, \dots, k, \dots)$, since $x \succ y$, from (2) there exists $n \in \mathbb{N}$ such that $(y_1, \dots, y_n, k, \dots, k, \dots) \prec x$, a contradiction which completes the proof. $\blacksquare$

The following corollary thus straightforwardly characterizes a large class of preferences for which one can guarantee the existence of IRPO allocations.

COROLLARY 1: *If all the preferences $\succsim_i \ i = 1, \dots, m$ of the decision makers satisfy A1 and are upper semi-myopic then there exist individually rational Pareto efficient allocations.*

PROOF: Immediate from Theorem 1 and Proposition 1.

Building upon Proposition 1, we give now some examples of weak* sequentially upper-semicontinuous preferences.

4 Weak* sequentially upper-semicontinuous preferences

In order to model preferences under uncertainty or even preferences on flow of payments it has been suggested (see Schmeidler, D. (1989) for the first circumstance and Gilboa, I. (1989) or else Chateauneuf and Rébillé (2004) in the second case) to use the Choquet integral (Choquet, 6.(1953)).

In our framework every $x \in \ell_+^\infty$ would be valued through $I(x) = \int x d\vartheta$ where ϑ is a capacity on $\mathcal{A} = 2^\mathbb{N}$ i.e. a set-function $\vartheta: A \in \mathcal{A} \rightarrow \vartheta(A) \in [0, 1]$ such that $\vartheta(\emptyset) = 0$, $\vartheta(\mathbb{N}) = 1$ and $A, B \in \mathcal{A}$. $A \subset B \Rightarrow \vartheta(A) \leq \vartheta(B)$ ³ and the Choquet integral of x w.r.t. to ϑ is defined as $\int x d\vartheta = \int_0^{+\infty} \vartheta(x \geq t) dt$.

It is well-know that preferences represented by the Choquet integral satisfy Assumption 1, therefore Choquet preferences $\succsim$ on ℓ_+^∞ are weak* sequentially u.s.c. iff $\succsim$ is upper semi-myopic.

It turns out that:

LEMMA 3: *A Choquet preference on ℓ_+^∞ is upper semi-myopic if and only if ϑ is outer-continuous, i.e. $\forall A_n, A \in \mathcal{A}, A_n \downarrow A \Rightarrow \vartheta(A_n) \downarrow \vartheta(A)$.*

PROOF: Outer-continuity is necessary. Actually let $A_n, A \in \mathcal{A}$ be such that $A_n \downarrow A$ and assume that $\vartheta(A_n) \downarrow k$ with $k > \vartheta(A)$. Therefore we get $k \mathbf{1}_\mathbb{N} \succ \mathbf{1}_A$.

Let $B_n = A_n \setminus A$, one gets $B_n \downarrow \emptyset$. So one can find an increasing sequence of integers $n(m)$, increasing with $m \in \mathbb{N}$ such that $\mathbf{1}_{E_m} \geq \mathbf{1}_{B_{n(m)}}$ where $E_m = \{p \in \mathbb{N}, p \geq m\}$. Under upper semi-myopia $\exists m_0$ such that $k \mathbf{1}_\mathbb{N} \succ \mathbf{1}_A + \mathbf{1}_{E_{m_0}}$ but $\mathbf{1}_{E_{m_0}} \geq \mathbf{1}_{B_{n(m_0)}} \Rightarrow \mathbf{1}_A + \mathbf{1}_{E_{m_0}} \geq \mathbf{1}_A + \mathbf{1}_{A_{n(m_0)} \setminus A} = \mathbf{1}_{A_{n(m_0)}}$. Hence by monotonicity of the Choquet integral one would get $k \mathbf{1}_\mathbb{N} \succ \mathbf{1}_{A_{n(m_0)}}$ but $I(\mathbf{1}_{A_{n(m_0)}}) = \vartheta(A_{n(m_0)}) \geq k$ and $I(k \mathbf{1}_\mathbb{N}) = k$, a contradiction.

Let us prove now that outer-continuity of ϑ is a sufficient condition.

³under uncertainty $\vartheta(A)$ is the subjective evaluation of the likelihood of the event A , while when valuing flow of payments, $\vartheta(A)$ is the weight given by the DM to the time period A .

It is enough to see that $\forall y, z \in \ell_+^\infty$, one gets $I(y + z_{E_n}) \rightarrow I(y)$ when $n \rightarrow +\infty$ one has:

$$I(y + z_{E_n}) = \int_0^{+\infty} \vartheta(y + z_{E_n} \geq t) dt.$$

Let $A_n(t) = \{y + z_{E_n} \geq t\}$ and $A(t) = \{y \geq t\}$. It is straightforward to see that $A_n(t) \downarrow A(t) \forall t \in \mathbb{R}_+$. From outer-continuity of ϑ , it turns out that $\forall t \in \mathbb{R}_+$, $f_n(t) = \vartheta(A_n(t)) \downarrow f(t) = \vartheta(A(t))$.

Thus by the Dominated Convergence Theorem $I(y + z_{E_n}) \rightarrow I(y)$.

So from the remark above we get:

PROPOSITION 2: *A Choquet preference on ℓ_+^∞ is weak* sequentially upper semi-continuous if and only if ϑ is outer-continuous i.e. $\forall A_n, A \in \mathcal{A}, A_n \downarrow A \Rightarrow \vartheta(A_n) \downarrow \vartheta(A)$.*

Imagine now that we are in situation of risk i.e. there is a given σ -additive probability P on $(\mathbb{N}, 2^\mathbb{N})$, with $P(\{n\}) > 0 \forall n$. Let us first consider Yaari's preference (Yaari (1987)), that is any $x \in \ell_+^\infty$ is valued through $I(x) = \int x d\vartheta$ where $\vartheta = f \circ P$ and f the distortion function: $f: [0, 1] \rightarrow [0, 1]$ satisfies $f(0) = 0$, $f(1) = 1$ is strictly increasing, and continuous.

From Proposition 2, one gets immediately:

PROPOSITION 3: *A preference à la Yaari on ℓ_+^∞ is weak* sequentially upper semi-continuous.*

Let us finally consider the classical expected utility preferences under risk.

In such a case $I(x) = \sum_{n=0}^{+\infty} P(\{n\})u(x(n))$ where u is assumed to be strictly increasing and continuous. It is immediate to check that such preferences satisfy Assumption 1.

In fact we obtain:

PROPOSITION 4: *A EU (expected utility) preference on ℓ_+^∞ is weak* sequentially upper semi-continuous.*

PROOF: From Proposition 1, it is enough to see that $\forall y, z \in \ell_+^\infty$ one gets $I(y + z_{E_n}) \rightarrow I(y)$ when $n \rightarrow +\infty$

$$I(y + z_{E_n}) = \sum_{m=0}^n P(\{m\})u(y(m)) + \sum_{m=n+1}^{+\infty} P(\{m\})u(y(m) + z(m)).$$

Since $y + z$ is norm bounded and u is monotone, it turns out that $u(y + z)$ is bounded, and since $\sum_{n=1}^{+\infty} P(\{m\}) \rightarrow 0$ when $n \rightarrow +\infty$ one gets $I(y + z_{E_n}) \rightarrow \sum_{n=0}^{+\infty} P(\{n\})u(y_n) = I(y)$.
■

Let us study now the important case of optimistic and pessimistic DM.

5 Optimistic and pessimistic DM

DEFINITION: A DM will be said pessimistic if her preferences are convex i.e. $\forall x, y \in \ell_+^\infty, \alpha \in (0, 1) \ x \succsim y \Rightarrow \alpha x + (1 - \alpha)y \succsim y$.

DEFINITION: A DM will be said optimistic if her preferences are concave i.e. $\forall x, y \in \ell_+^\infty, \alpha \in (0, 1) \ x \succsim y \Rightarrow x \succsim \alpha x + (1 - \alpha)y$.

Note that pessimism expresses that smoothing payoffs can only make the DM better off, while for an optimistic DM this will never be the case.

Indeed under uncertainty convexity of preferences is usually called uncertainty aversion (Schmeidler (89), Gilboa and Schmeidler (89)) while concavity is called uncertainty loving (see also Wakker (1990) and the seminal survey of Gilboa and Marinacci (2011)). It is well-known that for Choquet preferences, optimism is equivalent to a convex capacity (pessimism equivalent to a concave capacity), for Yaari preferences these two notions are respectively equivalent to f convex and f concave, and that for EU DM this amounts to respectively a concave or a convex VNM utility function u .

Let us recall:

DEFINITION: A capacity ϑ is convex (resp. concave) if $\forall A, B \in \mathcal{A} \ \vartheta(A \cup B) + \vartheta(A \cap B) \geq \vartheta(A) + \vartheta(B)$ (resp: $\vartheta(A \cup B) + \vartheta(A \cap B) \leq \vartheta(A) + \vartheta(B)$).

Let us recall also that for Choquet preferences optimism and pessimism are well motivated by the following properties (Schmeidler (86), (89)):

$$\text{If } \vartheta \text{ convex } \int x d\vartheta = \min\{Ep(x) | P \text{ additive probability } \geq \vartheta\}.$$

$$\text{If } \vartheta \text{ concave } \int x d\vartheta = \max\{Ep(x) | P \text{ additive probability } \leq \vartheta\}.$$

5.1 Existence of IRPO in presence of upper semi myopic pessimistic and reasonably optimistic DM

DEFINITION: An optimistic DM is said to be said reasonably optimistic if she is optimistic and her preference satisfies A1 and upper semi-myopia.

From the previous developments, the following DMs belong to the class of reasonable optimists:

- 1) Any optimistic EU (see Proposition 4)
- 2) Any optimistic Choquet DM such that $A_n, A \in \mathcal{A}, A_n \downarrow A \Rightarrow \vartheta(A_n) \downarrow \vartheta(A)$ (see Proposition 2).

Such DM exists it is enough to define $\vartheta = f \circ P$ where P σ -additive on $(\mathbb{N}, 2^{\mathbb{N}})$, f concave and right-continuous, but many other less specific examples of such capacities could be exhibited

- 3) Any optimistic Yaari's DM.

The terminology reasonably optimistic can be justified for a Choquet optimistic DM by the fact that it merely imposes that if an event decreases towards the empty set she reasonably appreciates this event less and less and is valued 0 at the limit.

LEMMA 4: *For a concave capacity the following assertions, are equivalent:*

- (i) $A_n, A \in \mathcal{A}, A_n \downarrow A \Rightarrow \vartheta(A_n) \downarrow \vartheta(A)$
- (ii) $A_n \in \mathcal{A} \ A_n \downarrow \phi \Rightarrow \vartheta(A_n) \downarrow 0$

PROOF: Such a property is a direct consequence of a dual similar property proved by Rosenmüller (1971) for the dual case of convex capacities.

So trivially, from Corollary 1 of Section 3, one gets:

PROPOSITION 5: *If all decision makers have preferences satisfying A1 and are either upper semi-myopic pessimistic or reasonably optimistic, then individually rational Pareto optimum exists.*

5.2 IRPO can fail to exist in the presence of a too optimistic DM

The next example shows an economy for which the unique PO is the trivial one that assigns the whole aggregate endowment for the DM 1.

In this example the nonexistence of an IRPO comes from the presence of an unreasonably optimistic DM. Namely DM₂ exhibits preferences represented by the Choquet integral $U^2(x) = \int x d\vartheta$ where ϑ is the concave capacity defined by $\vartheta(A) = 1 \ \forall A \neq \emptyset$, so clearly condition (ii) of Lemma 4 is not satisfied. Indeed since $U^2(x) = \sup_{s \in \mathbb{N}} x(s)$, such a DM is “terribly” optimistic.

EXAMPLE 1: NONEXISTENCE OF IRPO

Consider the consumption space $X = \ell_+^\infty$ and the probability measure p on $2^\mathbb{N}$ given by $p_s = \frac{1}{2^s} \ \forall s \in \mathbb{N}$.

- The decision maker 1 is characterized by the utility function $U^1(x) = \sum_{s \in \mathbb{N}} x_s p_s$ and the initial endowments $\omega^1 = (2 - \frac{1}{s})_{s \in \mathbb{N}}$;
- The decision maker 2 is characterized by the utility function $U^2(x) = \sup_{s \in \mathbb{N}} x_s$ and the initial endowments $\omega^2 = (2 - \frac{1}{s})_{s \in \mathbb{N}}$;

AFFIRMATION: The unique Pareto optimal allocation is the couple $(\bar{x}^1, \bar{x}^2)$ that associates the consumption plan $\bar{x}^1 = \omega^1 + \omega^2$ to DM 1 and $\bar{x}^2 = 0$ to DM 2. Thus, there is no IRPO.

In fact, consider a feasible allocation (x^1, x^2) with $x^2 > 0$.

If $\sup_{s \in \mathbb{N}} x_s^2 < \sup_{s \in \mathbb{N}} (\omega_s^1 + \omega_s^2) = 4$, take $s_0 \in \mathbb{N}$ be such that $x_{s_0}^2 > 0$. There is $s_1 \in \mathbb{N}$ large enough such that $U^1(x^1 + x_{s_0}^2 e_{s_0} - x_{E_{s_1}}^1) > U^1(x^1)$. As $U^2(x^2 - x_{s_0}^2 e_{s_0} + x_{E_{s_1}}^1) = 4 > U^2(x^2)$, we have that (x^1, x^2) is not a PO allocation.

If $\sup_{s \in \mathbb{N}} x_s^2 = 4$, given $n \in \mathbb{N}$ with $x_n^2 > 0$, we would get $U^1(x^1 + x_n^2 e_n) > U^1(x^1)$ and $U^2(x^2 - x_n^2 e_n) = U^2(x^2)$, so, again, (x^1, x^2) is not a PO allocation.

Thus, the unique PO allocation assigns $x^2 = 0$.

5.3 At an IRPO at most one strict optimistic will avoid ruin

DEFINITION: A DM is strictly optimistic if $\forall x, y \in \ell_+^\infty$, $x \neq y$ and $\forall \alpha \in (0, 1)$
 $x \succsim y \Rightarrow \alpha x + (1 - \alpha)y \prec x$ ⁴

In this section when we consider a strictly optimistic DM, we will assume that her preference $\succsim$ satisfies A1 but with the requirement of strict monotonicity that is $x, y \in \ell_+^\infty$
 $x > y$ (i.e. $x \geq y, x \neq y$) $\Rightarrow x \succ y$. It is easy to find examples of such strictly optimistic DM, who would be reasonably optimistic.

Indeed an EU DM as defined above in Section 4 with a strictly convex VNM utility would be such an example. Equally a Yaari's DM (see Section 4) with a strictly concave distortion function f would satisfy such a requirement.

In the Appendix C one will find the following Lemma 5 which characterizes such strict optimistic Choquet preferences.

LEMMA 5: *Choquet preferences on ℓ_+^∞ are strictly monotone, strictly optimistic and reasonably optimistic if and only if*

- (a) ϑ is strictly monotone i.e. $A \supsetneq B \Rightarrow \vartheta(A) > \vartheta(B)$
- (b) ϑ is strictly concave i.e. concave and such that:
 $A \cup B \supsetneq A \supsetneq B \supsetneq A \cap B \quad \vartheta(A \cup B) + \vartheta(A \cap B) < \vartheta(A) + \vartheta(B)$
- (c) ϑ is outer continuous.

THEOREM 2: *Consider an individually rational Pareto optimum where two DM i and j satisfy A1, have strictly optimistic preferences, weak* upper semi-continuous, and strictly monotonic then their allocations x^i and x^j satisfy for any $(s, t) \in \mathbb{N}^2$ $s \neq t$:*
 $x^i(s) \cdot x^i(t) \cdot x^j(s) \cdot x^j(t) = 0$.

PROOF: Let us define $x_{-(s,t)}^i$ by $x_{-(s,t)}^i(u) = \begin{cases} x^i(u) & \text{if } u \in \mathbb{N} \setminus \{s, t\} \\ 0 & \text{if } u \in \{s, t\} \end{cases} \quad e_s \in \ell_+^\infty$ by

⁴Indeed such a DM has non-convex preferences, but actually concave preferences.

$$e_s(u) = \begin{cases} 1 & \text{if } u = s \\ 0 & \text{otherwise} \end{cases} \quad \text{and consider } \forall (a, b) \in \mathbb{R}_+^2 \quad x_{(a,b)}^i \equiv a e(s) + b e(t) + x_{-(s,t)}^i.$$

Denote $\nu(s) \equiv x^i(s) + x^j(s)$ and $\nu(t) \equiv x^i(t) + x^j(t)$.

Therefore we define $x^j(a, b) \equiv (\nu(s) - a)e_s + (\nu(t) - b)e_t + x_{-(s,t)}^j$

Let us show that $x^i(s) > 0$, $x^i(t) > 0$, $x^j(s) > 0$, $x^j(t) > 0$ is impossible.

For any $a \in [0, x^i(s))$ by continuity of $\succsim_i$ and strict monotonicity of $\succsim_i$ $\exists b(a) > x^i(t)$ such that $x_{(a,b(a))}^i \sim_i x^i$. It is straightforward to check that when $a \uparrow x^i(s)$, then $b(a) \downarrow x^i(t)$. Therefore for $a = \bar{a}$ sufficiently close to $x^i(s)$, $\bar{b} = b(a)$ is such that $x_{(\bar{a}, \bar{b})}^j \geq 0$ i.e. $x_{(\bar{a}, \bar{b})}^j \in \ell_+^\infty$. That $x_{(\bar{a}, \bar{b})}^i \in \ell_+^\infty$ is immediate.

Let us set $\bar{y}_i = x_{(\bar{a}, \bar{b})}^i$ and $\bar{y}_j = x_{(\bar{a}, \bar{b})}^j$. Clearly $\bar{y}_i + \bar{y}_j = x^i + x^j$.

Since $\bar{y}_i \sim_i x^i$ if $\bar{y}_j \succ_j x^j$ IRPO would be contradicted. So necessarily $x^j \succsim_j \bar{y}_j$.

Let us choose $\beta \in (1, 2)$ close enough to 1 in order to guarantee that $y^i \equiv \beta x^i + (1 - \beta)\bar{y}_i$ and $y^j \equiv \beta x^j + (1 - \beta)\bar{y}_j$ are such that $y^i, y^j \geq 0$ (this is possible since by hypothesis $x^i(s) > 0$, $x^i(t) > 0$, $x^j(s) > 0$, $x^j(t) > 0$).

So y^i and y^j belong to ℓ_+^∞ and clearly $y^i + y^j = x^i + x^j$.

It is enough to prove that $y^i \succsim_i x^i$ and $y^j \succ_j x^j$ which contradicts again IRPO, and then completes the proof. One has

$$\begin{aligned} x^i &= \frac{1}{\beta} y^i + \frac{\beta - 1}{\beta} \bar{y}_i \text{ with } y^i \text{ and } \bar{y}_i \in \ell_+^\infty \\ x^j &= \frac{1}{\beta} y^j + \frac{\beta - 1}{\beta} \bar{y}_j \text{ with } y^j \text{ and } \bar{y}_j \in \ell_+^\infty \end{aligned}$$

Let us show that $y^i \succsim_i x^i$.

Otherwise $x^i \succ_i y^i$ hence $x^i \sim_i \bar{y}_i$ implies $\bar{y}_i \succ_i y_i$ and strict optimism implies $\bar{y}_i \succ_i x_i$ i.e. $x^i \succ_i x^i$, a contradiction.

Let us now show that $y^j \succ_j x^j$.

Suppose $x^j \succsim_j y^j$, note that $\bar{y}_j \neq y_j$ otherwise $y^j = \beta x^j + (1 - \beta)\bar{y}_j$ would imply $\bar{y}_j = x^j$ and this is not the case.

So either $y^j \succsim_j \bar{y}_j$ and $y^j \neq \bar{y}_j \Rightarrow y^j \succ_j x^j$ by strict optimism hence $x^j \succ_j x^j$, which is impossible. Or $\bar{y}_j \succ_j y^j$ and strict optimism would imply $\bar{y}_j \succ_j x^j$ which contradicts $x^j \succsim_j \bar{y}_j$. Henceforth $y^j \succ_j x^j$ which completes this part of proof.

Therefore we obtain the second main result of this paper.

COROLLARY 2: *At an IRPO at most one strict optimistic will avoid ruin in any state (or equally at any time period).*

PROOF: Assume that at least two DM i and j are strictly optimistic and satisfy the natural hypotheses of Theorem 2.

Imagine that one of them let us say i has an interior allocation, then at most in one state s the DM j can get $x^j(s) > 0$. ■

5.4 Optimal risk sharing for risk lovers and risk averters decision makers

Interpret ℓ^∞ as the set of all bounded $\mathcal{A}$ -measurable mappings x from $\mathbb{N}$ to $\mathbb{R}$ where $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and assume that a σ -additive probability P is given on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.⁵ So any x can now be interpreted as a random variable.

DEFINITION: Given $x, y \in \ell^\infty$, x is less risky than y for the second order stochastic dominance denoted $x \preceq_{SSD} y$ if:

$$\int_{-\infty}^t P(x \leq u) du \leq \int_{-\infty}^t P(y \leq u) du \quad \forall t \in \mathbb{R}$$

x is strictly less risky than y denoted $x \succ_{SSD} y$ if furthermore one of the previous inequalities is strict.

DEFINITION: Let $\preceq_i$ be the preference relation of a DM

$\preceq_i$ is a strict (strong) risk averter if $x \preceq_{SSD} y \Rightarrow x \preceq_i y$ and $x \succ_{SSD} y \Rightarrow x \succ_i y$

$\preceq_i$ is a strict (strong) risk lover if $x \preceq_{SSD} y \Rightarrow x \preceq_i y$ and $x \succ_{SSD} y \Rightarrow x \prec_i y$.

Examples of weak* sequentially upper-semicontinuous strict risk lover or strict risk averter

EU DM are such strict risk lovers or strict risk averters if and only if they are strictly optimistic or strictly pessimistic.

⁵For sake of simplicity we will assume that $P(\{n\}) > 0 \quad \forall n \in \mathbb{N}$.

Yaari's DM are such strict risk lovers or strict risk averters if and only if they are strictly optimistic or strictly pessimistic.

The following Theorem 3 offers two important features of risk sharing between strict risk averters and strict risk lovers.

Feature 1: As in the classical case of risk sharing between only risk averters, the Pareto optimal allocations of risk averters will be comonotonic, but note that this does not entail any longer that these allocations will be necessarily ranked as the aggregate endowment, because nothing in the Theorem guarantees global comonotonicity between risk averters and risk lovers.

Feature 2: As soon as the strict risk lovers are also strictly optimistic (note that while this is the case for EU or Yaari's DM, this might not be the case for some particular class of strict risk lovers (see Chateauneuf and Lakhnati (2007)) then at most one of these strict risk lovers will avoid ruin.

THEOREM 3: *Consider m strict risk averters $i = 1, \dots, m$ with weak* sequentially u.s.c. preferences $\succsim_i$ and n strict risk lovers $j = 1, \dots, n$ with weak* sequentially u.s.c. preferences $\succsim_j$ but exhibiting strict optimism.*

Assume initial endowments $w_i \in \ell_{++}^\infty$, $w_j \in \ell_{++}^\infty$.

Then individual rational Pareto efficient allocations (in $(\ell_+^\infty)^{m+n}$) exist. For such PO $(x_i, i = 1, \dots, m; y_j, j = 1, \dots, n)$ we have:

- 1) *The allocations of risk averters are pairwise comonotonic, i.e.,*

$$(x_{i_1}(s) - x_{i_1}(t))(x_{i_2}(s) - x_{i_2}(t)) \geq 0 \quad \forall (s, t) \in \mathbb{N}^2, \forall (i_1, i_2) \in \{1, \dots, m\}^2$$
- 2) *At most one risk lover will avoid ruin in any state.*

PROOF:

- 1) The same proof as in Proposition 4.1 of Chateauneuf et al. (2000) applies.
- 2) This is just Corollary 2.

6 Conclusions and Further Research

As Theorem 1 has shown, the existence of IRPO allocation is a quite general feature of GE models when the space of alternatives is ℓ^∞ (or $\mathbb{R}^S$), even in the presence of non-complete preferences or optimistic agents. One of the sufficient requirements is that the DMs should be myopic or impatient for gains, which can be interpreted as a bound on optimism, since, in that case, even the optimistic DMs will not want to exchange all their wealth for a larger income on events with arbitrarily small likelihood or dates arbitrarily far (which avoids the difficulties present in Example 1).

So, one perceives that the concept of IRPO accommodates well the notions of risk and ambiguity propensity (both relevant for Applied Microeconomics and Finance). A further research program is to find the sufficient conditions to ensure the existence of a barter equilibrium (i.e., the non-emptiness of the core of the economy) or an equilibrium with non-linear prices even in the presence of optimistic agents.

At an IRPO allocation of resources, the risk averters will be sharing the individual risk (in the sense that their plans will be 2×2 comonotonic), but not necessarily the aggregate risk when there exist risk lovers DMs too. However, when the aggregate risk is very large, the existence of optimistic agents may allow the risk averters to be better off than they would be sharing aggregate risk as it becomes possible to smooth their consumption plans.

Regarding the properties of IRPO allocation, Theorems 2 and 3 say that is quite likely to observe plans of optimistic DMs leading to financial ruin at some states. In fact, if there were two optimistic DMs with consumption plans that are bounded away from zero in two states of nature, there would be room for a Pareto improvement. Under the theorems hypotheses, in the case that the consumption plan is interior for one of the optimistic DMs, all the others will concentrate the future wealth in just one of the possible future states, characterizing a very speculative pattern. Thus, a question that naturally arises for future studies is if a policy maker concerned with the ex post aggregate welfare should or not should deviate the output allocation of the economy from ex ante IRPO allocations in order to avoid this extreme gambling. And, if the policy maker decides to intervene, which type of instrument should be chosen. Or even, if the incompleteness of markets

could weaken the effects of this willingness to gamble future wealth, thereby preventing any need of intervention.

APPENDIX

A The space ℓ^∞

The space ℓ^∞ is the Banach space of real bounded sequences equipped with the norm defined by $\|x\| = \sup_t |x_t|$. The space ℓ^1 is the Banach space set of absolutely convergent real sequences equipped with the norm defined by $\|x\|_1 = \sum_{t=1}^{\infty} |x_t|$.

When ℓ^∞ is endowed with the norm topology, the dual is denoted by $(\ell^\infty)^*$. A coarser topology is the *Mackey topology*, defined as the finest topology on ℓ^∞ for which the dual is ℓ^1 . Now, (x^n) converges to x in this topology if and only if, for any weakly compact subset A of ℓ^1 , $\langle x^n, y \rangle \rightarrow \langle x, y \rangle$ uniformly on $y \in A$.

The weak* topology is defined as the coarsest topology on ℓ^∞ for which the dual is ℓ^1 . Now (x^n) converges to x in the weak* topology if and only if $\|x^n\|$ is bounded and $x^n(p) \rightarrow x(p) \ \forall p \in \mathbb{N}$ (see for instance Brezis (2011)).

B Mackey and weak* sequential convergence coincide

ON ℓ^∞

PROPOSITION 6: *On ℓ^∞ the weak* $\sigma(\ell^\infty, \ell^1)$ sequential convergence and the Mackey $\tau(\ell^\infty, \ell^1)$ sequential convergence coincide.*

PROOF: Since $\sigma(\ell^\infty, \ell^1) \subset \tau(\ell^\infty, \ell^1)$ we just need to prove that for (x^n) , $x^n \in \ell^\infty$, $x^n \xrightarrow{w^*} x$ implies $x^n \xrightarrow{\tau} x$. But from the corollary in the appendix in Hervés-Beloso et

al (JET, 2000), we know that the Mackey topology and the weak star topology coincide on bounded sets of ℓ^∞ .

Since $x^n \xrightarrow{w^*} x$ implies that the sequence (x^n) is bounded, the result follows directly. ■

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Here we intend to prove Lemma 5.

(b) can be found in Schmeidler (1989)

(c) comes from Proposition 2.

The necessity of (a) is immediate.

It remains to prove that under (a) and (c) the preferences are strictly monotone.

So take $x, x' \in \ell_+^\infty$ with $x > x'$ and let us show that $x \succ x'$.

$x > x'$ implies there exists $s_0 \in \mathbb{N}$ s.t. $x(s_0) > x'(s_0)$. We need to show that

$$\int_0^{+\infty} \vartheta(x \geq t) dt > \int_0^{+\infty} \vartheta(x' \geq t) dt$$

or equally $d > 0$ where

$$d = \int_0^{+\infty} (\vartheta(x \geq t) - \vartheta(x' \geq t)) dt.$$

Let $t_0 = x(s_0)$, therefore $\{x \geq t_0\} \supsetneq \{x' \geq t_0\}$ and (a) implies $\vartheta(x \geq t_0) > \vartheta(x' \geq t_0)$.

Let $y \in \ell_+^\infty$, and let us see that $f(t) = \vartheta(y \geq t)$ is left-continuous.

Take $A_n = \{s \in \mathbb{N}, y(s) \geq t_n\}$ with $t_n \in \mathbb{R}^+$, $t_n \uparrow t$. It is straightforward to see that $A_n \downarrow A = \{y \geq t\}$, so since from (c) $\vartheta(A_n) \downarrow \vartheta(A)$, f is actually left-continuous. So since $t_0 = x(s_0) > 0$, there exists an interval $[t_0 - \varepsilon, t_0] \subseteq \mathbb{R}^+$ such that $\forall t \in [t_0 - \varepsilon, t_0]$, $\vartheta(x \geq t) - \vartheta(x' \geq t) > 0$, since indeed $x \geq x'$ implies $\vartheta(x \geq u) \geq \vartheta(x' \geq u) \forall u \in \mathbb{R}^+$ one gets $d > 0$. ■

REFERENCES

- ALIPRANTIS, C., AND K.C. BORDER (1999): *A course in functional analysis*, Springer-Verlag, New York.
- ALIPRANTIS, C., AND BURKINSHAW (1978): *Locally solid Riesz spaces*, Pure and applied mathematics. Academic Press.
- ARAÚJO, A. (1985): “Lack of Pareto Optimal Allocations in Economies with Infinitely Many Commodities: The Need for Impatience”, *Econometrica*, 53(2), 455-61.
- ARAÚJO, A., R. NOVINSKI AND M.R. PÁSCOA (2011): “General equilibrium, wariness and efficient bubbles”, *Journal of Economic Theory*, 146(3), 785-811.
- AZÉ, D. AND J.-B. HIRIART-URRUTY (2010): *Analyse variationnelle et optimisation*. Toulouse: Cépaduès-Éditions.
- BEWLEY, T.F. (1972): “Existence of equilibria in economies with infinitely many commodities”, *Journal of Economic Theory*, 4(3), 514-540.
- BREZIS, H., (2011): *Functional analysis, Sobolev spaces and partial differential equations*, Springer.
- BROWN, D.J. AND L.M. LEWIS (1981): “Myopic Economic Agents”, *Econometrica*, 49(2), 359-68.
- CHATEAUNEUF, A. AND Y. RÉBILLÉ (2004): “Some characterizations of non-additive multi-period models”, *Mathematical Social Sciences*, 48(3), 235-250.
- CHATEAUNEUF, A.J., R.-A. DANA AND J.-M. TALLON (2000): “Optimal risksharing rules and equilibria with Choquet-expected-utility”, *Journal of Mathematical Economics*, 34(2), 191-214.
- CHATEAUNEUF, A.J. AND G. LAKHNATI (2007): “From sure to strong diversification”, *Economic Theory*, 32, 511-522.

- CHOQUET, G., (1953): “Théorie des Capacités”, *Ann. Inst. Fourier (Grenoble)* V, 131-295.
- CONWAY, J.B., (1985): *A course in functional analysis*, Springer-Verlag, New York.
- GILBOA, I., (1989): “Expectation and variation in multi-period decisions”, *Econometrica*, 57, 1153-1169.
- GILBOA, I. AND M. MARINACCI, (2012): “Ambiguity and the Bayesian paradigm”, In: *Advances in Economic and Econometrics: Theory and Applications*, Ten World Congress of the Econometric Society.
- GILBOA, I. AND D. SCHMEIDLER, (1989): “Maxmin expected utility with a non-unique prior”, *Journal of Mathematical Economics*, 110, 605-639.
- HERVÉS-BELOSÓ, C., MORENO-GARCIA, E., NÚÑEZ-SAUZ, C. AND PÁSCOA, M.R. (2000): “Blocking efficacy of small coalitions in myopic economics”, *Journal of Economic Theory*, 93(1), 72-80.
- ROSENMÜLLER, J. (1972): “Some properties of convex set functions”, *Methods of Operations Research*, 17, 277-307.
- SCHMEIDLER, D. (1986): “Integral representation without additivity”, *Proceedings of the American Mathematical Society*, Vol. 97, No. 2.
- SCHMEIDLER, D. (1989): “Subjective Probability and Expected Utility without Additivity”, *Econometrica*, 57(3), 571-87.
- WAKKER, P.P. (1990): “Characterizing optimism and pessimism directly through comonotonicity”, *Journal of Economic Theory* 5, 453-463.