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Frédéric Godin, Van Son Lai, Denis-Alexandre Trottier

IPAG BUSINESS SCHOOL

184, Boulevard Saint-Germain
75006 Paris
France

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A general class of distortion operators for pricing contingent claims with applications to CAT bonds*

Frédéric Godin[a,b,c], Van Son Lai[d,e], and Denis-Alexandre Trottier[d]

[a]Concordia University, Department of Mathematics and Statistics, Montréal (Québec), Canada
[b]Université Laval, École d'Actuariat, Québec (Québec), Canada
[c]Quantact Actuarial and Financial Mathematics Laboratory, Montréal (Québec), Canada
[d]Université Laval, Faculté des Sciences de l'Administration, Québec (Québec), Canada
[e]IPAG Business School, Paris, France

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Email addresses: frederic.godin@concordia.ca (Frédéric Godin), VanSon.Lai@fsa.ulaval.ca (Van Son Lai) and denis-alexandre.trottier.1@ulaval.ca (Denis-Alexandre Trottier).

A general class of distortion operators for pricing contingent claims with applications to CAT bonds

Abstract

The current paper provides a general approach to construct distortion operators that can price financial and insurance risks. Our approach generalizes the Wang (2000) transform and recovers multiple distortions proposed in the literature as particular cases. This approach enables designing distortions that are consistent with various pricing principles used in finance and insurance such as no-arbitrage models, equilibrium models and actuarial premium calculation principles. Such distortions allow for the incorporation of risk-aversion, distribution features (e.g., skewness and kurtosis) and other considerations that are relevant to price contingent claims. The pricing performance of multiple distortions obtained through our approach is assessed on CAT bonds data. The current paper is the first to provide evidence that jump-diffusion models are appropriate for CAT bonds pricing, and that natural disaster aversion impacts empirical prices. A simpler distortion based on a distribution mixture is finally proposed for CAT bonds pricing to facilitate the implementation.

Keywords: Distortion operator, Wang transform, Distortion risk measure, Arbitrage-free pricing, Insurance pricing, Contingent claim pricing, CAT bonds.

1 Introduction

In his seminal paper, Wang (2000) proposes a probability distortion operator based on the Gaussian distribution to price both financial and insurance risks. In particular, the latter distortion is shown to allow recovering the classical Black-Scholes option pricing formula. Among others, Hamada and Sherris (2003) and Pelsser (2008) study the applicability of the Wang transform to contingent claims pricing. They find that the associated normality assumption poses significant limitations in practical contexts often involving non-Gaussian distributions. Generalizations of the Wang transform handling non-normality were developed in the literature. For instance, Godin et al. (2012) develop a distortion based on the Normal Inverse Gaussian (NIG) distribution which allows recuperating arbitrage-free derivatives prices when the underlying asset is driven by a NIG Lévy process.

Our main contribution consists in proposing a general approach to generalize the Wang transform and construct a broad class of distortion operators. Multiple distortions proposed in the literature are included in this family as particular cases. Moreover, our approach enables designing distortions that are consistent with various pricing principles used in financial engineering and insurance such as no-arbitrage models, equilibrium models and actuarial premium calculation principles. In particular, using our methodology, we produce new distortions that can recuperate pricing functionals of popular financial and insurance models well beyond the Wang transform and its non-Gaussian extensions. The current paper offers a general description of the connection between distortion operators and other pricing principles. For instance, connections between the Wang transform, the Black-Scholes model, the Esscher-Girsanov change of measure and the Bühlmann's equilibrium are recovered as special cases of our analyses. This generalization therefore extends several literature papers studying such connections. Conditions under which the Bühlmann (1980)'s pricing equilibrium yields the Wang transform are derived by Wang (2003). Multivariate extensions of the Wang transform based on similar lines of reasoning can be found in Kijima (2006), Wang (2007), and Kijima and Muromachi (2008). The equivalence between the Wang transform and the Esscher-Girsanov change of measure proposed by Goovaerts and Laeven (2008)

is demonstrated in a static setting by Labuschagne and Offwood (2010).

The flexibility of the approach presented in this paper not only allows recuperating existing distortions; it also enables easily designing new ones with advantageous properties that are suited to price both financial and insurance risk. In particular, several novel distortions are provided in the current work such as a NIG Esscher transform distortion extending the work of Godin et al. (2012), a jump-diffusion distortion based on the Kou (2002) model, a distortion consistent with size-biased premiums for generalized beta of the second kind (GB2) distribution and a distortion consistent with the Esscher pricing principle for the gamma distribution.

Another potential application of the approach provided in the current paper is the production of new *distortion risk measures*, which are quantile-based measures used by practitioners in finance and insurance, see Dowd and Blake (2006). An advantage of using our proposed distortion operators to design risk measures is that they easily allow for the incorporation of risk-aversion, distribution features (e.g., skewness and kurtosis) and other considerations that are embedded into the choice of a pricing functional. Several literature papers attempted incorporating these features into risk measures (e.g., Bali and Theodossiou, 2008; Gzyl and Mayoral, 2008).

The second contribution of this paper is to conduct an empirical assessment of the performance of various distortion operators obtained from our approach in terms of their ability to accurately depict catastrophe (CAT) bond market spreads. In our study, jump-diffusion distortions outperform other distortions based either on the Gaussian, the Student-t, the Normal Inverse Gaussian (NIG) and the generalized beta of the second kind (GB2) distributions. To the best of the authors knowledge, the current paper is the first to provide evidence that jump-diffusion models are appropriate for pricing CAT bonds. Moreover, calibration results offer evidence that jump risk is priced and thus that investors are averse to natural disasters. Finally, a simplified version of the jump-diffusion distortion based on a probability mixture is proposed to ease computations and facilitate implementation. This simpler version is shown to also produce good pricing performance.

The paper proceeds as follows. Section 2 provides some background on distortion operators and other forms of risk pricing. Section 3 presents the general approach proposed to construct

distortion operators and characterizes its applicability to arbitrage-free pricing. Section 4 derives new distortion operators consistent with popular financial non-Gaussian option pricing models. Section 5 performs the same exercise for insurance pricing models. Section 6 presents empirical applications to the pricing of catastrophe bonds. Section 7 concludes and discusses other potential applications.

2 Background on risk pricing

In this section, we briefly review popular financial and insurance pricing principles.

2.1 Arbitrage-free pricing

Consider a continuous-time economy where time t takes value within $[0, T]$. This economy is characterized by a probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions.

A central result in the theory of asset pricing by arbitrage, the so-called first fundamental theorem of asset pricing, is the equivalence between the absence of (quasi-)arbitrage opportunities and the existence of an equivalent (local) martingale measure $\mathbb{Q}$. See Delbaen and Schachermayer (1994) for the history of this theorem which goes back to the seminal papers of Harrison and Kreps (1979), and Harrison and Pliska (1981). Such a measure $\mathbb{Q}$ is often called a *risk-neutral measure* because under some technical conditions, discounted asset prices are genuine martingales and the arbitrage-free price process $\{S_t\}_{t\in[0,T]}$ of any traded asset or derivative must satisfy

$$S_t = B_t \, \mathbb{E}^{\mathbb{Q}}\left[\frac{S_s}{B_s}\middle|\mathcal{F}_t\right], \qquad \forall t \leq s \leq T, \tag{2.1}$$

where $\{B_t\}_{t\in[0,T]}$ is the risk-free asset price process.

(Re)insurance contracts (Delbaen and Haezendonck, 1989; Sondermann, 1991) and insurance-linked securities (Vaugirard, 2003) are often priced through an arbitrage-free approaches initially developed for financial derivatives. Such frameworks are often characterized by an underlying insurance or catastrophe loss process whose dynamics is punctuated by random jumps, e.g., a

jump-diffusion process. Even if the underlying asset is traded, the presence of jumps typically leads to market incompleteness, which entails the existence of an infinite number of possible contract prices. In this case, prices are determined by the market, i.e., by the equilibrium resulting from supply and demand which are in turn determined by aggregate risk-aversion, liquidity needs, and other factors. In the literature, pricing models under such contexts are often obtained either through the modeling of the equilibrium framework, or through the use of particular families of risk-neutral measures possessing convenient theoretical or computational properties such as the minimal martingale measure (Föllmer and Schweizer, 1991), the Esscher martingale measure (Gerber and Shiu, 1994), the variance optimal martingale measure (Schweizer, 1995) and the mean-correcting martingale measure (Schoutens, 2003).

2.1.1 Equilibrium-based martingale measures

General equilibrium models allow selecting a particular risk-neutral measure when the market is incomplete. For example, consider a continuous-time extension of Lucas (1978)'s model. Assume there is a representative agent entitled to an endowment who maximizes his aggregate utility. Let $U(t, c_t)$ be the aggregate utility at time t for his consumption process $\{c_t\}_{t\in[0,T]}$. Under mild conditions, one can show (see, e.g., Stokey and Lucas, 1989) that this setup produces a pricing kernel that depends only on the aggregate endowment process, denoted by $\{\delta_t\}_{t\in[0,T]}$, such that the price process $\{S_t\}_{t\in[0,T]}$ of any traded asset or derivative must satisfy the following condition in equilibrium:

$$S_t = \mathbb{E}^{\mathbb{P}}\left[\frac{U_c(s,\delta_s)}{U_c(t,\delta_t)}\, S_s \middle| \mathcal{F}_t\right], \qquad \forall t \leq s \leq T, \tag{2.2}$$

where $\mathbb{P}$ is the physical measure, and $U_c \equiv \frac{\partial U}{\partial c}$. This equilibrium condition can be written in the form of equation (2.1) by noting that the pricing kernel can be used to define a Radon-Nikodym derivative.

2.2 Actuarial premium calculation principles

A prominent problem in actuarial science is to derive premium calculation principles (PCPs) that satisfy a number of desirable properties (see, e.g., Laeven and Goovaerts, 2008). We present

below two of such popular approaches.[1]

2.2.1 Distortion operators

The history of distortion operators goes back to Yaari (1987)'s dual theory of choice under risk, in which attitudes toward risks are characterized by a distortion function rather than by an expected utility function. Distortion operators also stem from the axiomatic approach of Wang et al. (1997) to characterize insurance prices.

Let X be a random variable representing a financial risk. Denote its survival function under the physical measure $\mathbb{P}$ by $\bar{F}_{X,\mathbb{P}}(x) \equiv \mathbb{P}(X > x)$. A distortion operator g is an increasing and differentiable function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$.

The function g allows constructing a new survival function defined as $g(\bar{F}_{X,\mathbb{P}}(\cdot))$ by distorting the original survival function of X. The price of X is obtained as its expected value using the distorted survival function. One can show that this expected value has the following Choquet integral representation:

$$H[X; g] \equiv \int_{-\infty}^{0} \big[g\big(\bar{F}_{X,\mathbb{P}}(x)\big) - 1\big]dx + \int_{0}^{\infty} g\big(\bar{F}_{X,\mathbb{P}}(x)\big)dx. \tag{2.3}$$

This Choquet integral exhibits monotonicity, translation invariance, positive homogeneity, and is sub-additive if g is concave (Denneberg, 1994). Hence, the functional $H[\cdot; g]$ defines a distortion-based risk measure that is coherent in the sense of Artzner et al. (1999) if g is concave.[2]

2.2.2 Actuarial weighted pricing principles

Furman and Zitikis (2008) propose a broad class of PCPs based on weighted loss distributions. This class can also be viewed as a subclass of the loss function approach (see Remark 1 of Heilmann, 1989). In their approach, Furman and Zitikis (2008) introduce a weighting function w

[1] There is a plethora of other insurance pricing principles; the reader is referred to Laeven and Goovaerts (2008) and Ai and Brockett (2008) for thorough accounts of these.

[2] A more comprehensive treatise on distortion-based risk measures can be found in Dowd and Blake (2006).

such that $w(x) \geq 0$ for all $x \geq 0$. The price of any risk $X \geq 0$ is then defined by

$$\Pi[X; w] \equiv \begin{cases} \dfrac{\mathbb{E}^{\mathbb{P}}[w(X)X]}{\mathbb{E}^{\mathbb{P}}[w(X)]}, & \text{if } 0 < \mathbb{E}^{\mathbb{P}}[w(X)] < \infty, \\ 0, & \text{if } \mathbb{E}^{\mathbb{P}}[w(X)] = 0, \\ \infty, & \text{if } \mathbb{E}^{\mathbb{P}}[w(X)] = \infty. \end{cases} \tag{2.4}$$

Thus, this approach is characterized by using a risk specific pricing kernel $w(X)/\mathbb{E}^{\mathbb{P}}[w(X)]$ to price a risk X. Several popular PCPs can be obtained through this weighting procedure. For instance, the Esscher principle is obtained with $w(x) = e^{ax}$ and the size-biased premium principle with $w(x) = x^a$.[3]

3 A general framework for distortion-based valuation

The purpose of this section is to draw a parallel between no-arbitrage pricing and distortion-based pricing methods; this is done by showing how to construct a distortion operator yielding a price that is consistent with a given martingale measure for a wide class of contingent claims. First, the general definition of our distortion operator is presented. We then illustrate how this operator can be used to recover arbitrage-free prices of derivatives under various pricing models.

3.1 A new class of distortion operators

Assume the existence of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and of a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$. We first introduce a random variable whose survival function is invertible.

Definition 3.1. *Throughout this whole section, Z refers to an absolutely continuous random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ whose survival function defined through $\bar{F}_{Z,\mathbb{P}}(z) \equiv \mathbb{P}(Z > z)$ is strictly decreasing over $(ess\ inf\ Z, ess\ sup\, Z)$. This implies $\bar{F}_{Z,\mathbb{P}} : [ess\ inf\, Z, ess\ sup\, Z] \to [0, 1]$ is invertible, where $ess\ inf\ Z \in \mathbb{R} \cup \{-\infty\}$ and $ess\ sup\, Z \in \mathbb{R} \cup \{\infty\}$. Denote by $\bar{F}^{-1}_{Z,\mathbb{P}}$ its inverse. In particular, $\bar{F}^{-1}_{Z,\mathbb{P}}(1) = ess\ inf\ Z$ and $\bar{F}^{-1}_{Z,\mathbb{P}}(0) = ess\ sup\ Z$.*

[3] We refer the reader to Table 1 of Furman and Zitikis (2009) for additional examples.

We now define the function which will serve as the general distortion operator.

Definition 3.2. Define the function

$$g_Z^{\mathbb{Q},\mathbb{P}}(u) \equiv \bar{F}_{Z,\mathbb{Q}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(u)\big), \qquad u \in [0,1]. \tag{3.1}$$

Note that $g_Z^{\mathbb{Q},\mathbb{P}}$ is a genuine distortion operator as the absolute continuity of Z under both $\mathbb{P}$ and $\mathbb{Q}$ and the strict positivity of its probability density function almost everywhere over its support guarantee that $g_Z^{\mathbb{Q},\mathbb{P}}$ is both strictly increasing and differentiable. It might be possible to devise distortions using more general distributions which don't necessarily need to satisfy conditions from Definition 3.1, e.g., with atoms, by considering for instance Rüschendorf (2009)'s transform. However, we won't develop such generalizations in the current work as risk distributions used in practice typically have a strictly positive density over their support.

The following proposition shows that the distortion implied by a random variable Z is the same than the one implied by $h(Z)$ if Z and h satisfy a set of conditions. Although the proof of this result is straightforward, we include it for completeness in Appendix A.1.

Proposition 3.1. *Consider any continuous and strictly increasing function* $h : [\text{ess inf } Z, \text{ess sup } Z] \to \mathbb{R} \cup \{-\infty, \infty\}$ *such that* $h(-\infty) \equiv \lim_{x\to-\infty} h(x)$ *if* ess inf $Z = -\infty$ *and* $h(\infty) \equiv \lim_{x\to\infty} h(x)$ *if* ess sup $Z = \infty$. *Then,*

$$g_{h(Z)}^{\mathbb{Q},\mathbb{P}}(u) = g_Z^{\mathbb{Q},\mathbb{P}}(u), \qquad \forall u \in [0,1]. \tag{3.2}$$

Proposition 3.1 proves to be very useful as it can often allow simplifying the general expression (3.1) to a simpler form with fewer parameters. This can be crucial to circumvent parameter identification issues that could otherwise arise in calibration. Such simplifications are illustrated by the next remark.

Remark 3.1 (Wang transform)**.** Suppose that Z is a standard normal $N(0,1)$ random variable under the measure $\mathbb{P}$, and that it is shifted to a $N(\theta,1)$ distribution under the measure $\mathbb{Q}$. In other words, we have $\mathbb{P}(Z > z) = 1-\Phi(z)$, and $\mathbb{Q}(Z > z) = 1-\Phi(z-\theta)$. Let h be any continuous

and strictly increasing function, e.g., an exponential asset model $h(z) \equiv S_0 e^{\mu T + \sigma\sqrt{T}z}$ with $\sigma > 0$, $T > 0$ and $S_0 > 0$. From Proposition 3.1, we have $g^{\mathbb{Q},\mathbb{P}}_{h(Z)}(u) = g^{\mathbb{Q},\mathbb{P}}_{Z}(u) = \Phi\big(\Phi^{-1}(u) + \theta\big)$ which is a single-parameter distortion that does not depend on μ, σ, S_0, nor T.

3.2 Risk-neutral pricing using distortion operators

Let X be a random variable representing a financial or insurance risk. For example, X could be the price of a traded asset (as it often is for the case of financial derivatives), but it could also be something else like the value of a catastrophe loss index (as for insurance-linked securities). The goal is to price derivatives on X using a distortion operator.

In what follows, we assume $X = h(Z)$ where Z is a random variable driving the risk and h is a continuous increasing mapping. The Proposition 3.2 whose proof is in Appendix A.2 and the Corollary 3.1 below show that a price consistent with a given pricing measure $\mathbb{Q}$ (such as a risk-neutral measure) can be recovered by using the distortion operator $g^{\mathbb{Q},\mathbb{P}}_{Z}$.

Proposition 3.2. *Let $X = h(Z)$ where $h : \mathbb{R} \to \mathbb{R}$ is continuous and increasing. Then, the distorted $\mathbb{P}$-distribution of X coincides with its distribution under $\mathbb{Q}$:*

$$\bar{F}_{X,g^{\mathbb{Q},\mathbb{P}}_{Z}}(x) \equiv g^{\mathbb{Q},\mathbb{P}}_{Z}\big(\bar{F}_{X,\mathbb{P}}(x)\big) = \bar{F}_{X,\mathbb{Q}}(x), \text{ for all } x \in \mathbb{R}.$$

The following result is a direct consequence of applying Proposition 3.2 in (2.3).

Corollary 3.1. *For any continuous and increasing function $h : \mathbb{R} \to \mathbb{R}$ such that $h(Z)$ is $\mathbb{Q}$-integrable,*

$$H\Big[h(Z); g^{\mathbb{Q},\mathbb{P}}_{Z}\Big] = \mathbb{E}^{\mathbb{Q}}[h(Z)]. \tag{3.3}$$

In particular, Corollary 3.1 holds for any equivalent martingale measure $\mathbb{Q}$, in which case the arbitrage-free price of a derivative with terminal payoff $h(Z)$ is given by the discounted value[4] of $H\Big[h(Z); g^{\mathbb{Q},\mathbb{P}}_{Z}\Big]$. Particular cases of this result can be found in the literature. For instance, the proof when Z is the terminal value of a geometric Brownian motion (see Remark 3.3) can be

[4]Here we assume the numéraire is deterministic.

found in Hamada and Sherris (2003). The proof when Z is the terminal value of an exponential NIG Lévy process and $\mathbb{Q}$ is the mean-correcting martingale measure is given in Godin et al. (2012).

Remark 3.2. Although Proposition 3.1 requires h to be strictly increasing, Proposition 3.2 and Corollary 3.1 only require h to be increasing (i.e., non-decreasing), which is less stringent.

Remark 3.3 (Black-Scholes model)**.** Consider contingent claims of the form $h(S_T)$ where h is continuous and increasing and S_T is the time-T underlying asset price. It follows from Corollary 3.1 that the distortion operator that can recuperate their arbitrage-free prices under the Black-Scholes model is $g_{S_T}^{\mathbb{Q},\mathbb{P}}$, where S_T is the terminal value of a geometric Brownian motion, with constant drift μ and volatility σ under the physical measure $\mathbb{P}$, and with drift r and volatility σ under the risk-neutral measure $\mathbb{Q}$, where r is the risk-free rate. Following Remark 3.1, one can check that this distortion reduces to the Wang transform $g_{S_T}^{\mathbb{Q},\mathbb{P}}(u) = \Phi\big(\Phi^{-1}(u) + \varphi\big)$ where $\varphi = \big(\frac{r-\mu}{\sigma}\big)\sqrt{T}$, with T being the contingent claim maturity.

3.3 Connections with the pricing kernel approach

We now illustrate the connection between the use of the distortion operator $g_Z^{\mathbb{Q},\mathbb{P}}$ from Definition 3.2 and of pricing kernels. Such a connection is characterized below by Theorem 3.1 whose proof is in Appendix A.3.

Definition 3.3. Define $f_{Z,\mathbb{P}}$ as the density of Z under $\mathbb{P}$, and define $f_{Z,\mathbb{Q}}$ as its density under $\mathbb{Q}$. Define the following likelihood ratio:

$$\xi_Z^{\mathbb{Q},\mathbb{P}}(z) \equiv \frac{f_{Z,\mathbb{Q}}(z)}{f_{Z,\mathbb{P}}(z)}, \quad z \in (ess\ inf Z, ess\ sup Z). \tag{3.4}$$

Theorem 3.1. *Let X be an absolutely continuous random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. The density of X after distorting its survival function with $g_Z^{\mathbb{Q},\mathbb{P}}$, which is represented through $f_{X,g_Z^{\mathbb{Q},\mathbb{P}}}(x) \equiv$*

$\frac{\partial}{\partial x}\bar{F}_{X,g_Z^{\mathbb{Q},\mathbb{P}}}(x)$, *is given by*

$$f_{X,\,g_Z^{\mathbb{Q},\mathbb{P}}}(x) = f_{X,\mathbb{P}}(x)\,\xi_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(\bar{F}_{X,\mathbb{P}}(x))\big)\,, \quad x \in (ess\ inf X, ess\ sup X). \tag{3.5}$$

Remark 3.4. *The random variable* $\xi_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(\bar{F}_{X,\mathbb{P}}(X))\big)$ *could be used as a Radon-Nikodym derivative to generate a new probability measure, i.e., one could generate an equivalent measure* $\tilde{\mathbb{Q}}$ *through* $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \equiv \xi_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(\bar{F}_{X,\mathbb{P}}(X))\big)$. *The latter could thus be embedded in a pricing kernel, which would provide a price for* X *that is consistent with the one obtained through the distortion* $g_Z^{\mathbb{Q},\mathbb{P}}$, *since*

$$\mathbb{E}^{\tilde{\mathbb{Q}}}[X] = \mathbb{E}^{\mathbb{P}}\left[X\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\right] = \int_{\mathbb{R}} x f_{X,\mathbb{P}}(x)\,\xi_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(\bar{F}_{X,\mathbb{P}}(x))\big)\,dx = \int_{\mathbb{R}} x f_{X,\,g_Z^{\mathbb{Q},\mathbb{P}}}(x)dx.$$

Moreover, under the assumption that $X = h(Z)$ *for a continuous and strictly increasing mapping* h, *then* $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} = \xi_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(\bar{F}_{X,\mathbb{P}}(X))\big) = \xi_Z^{\mathbb{Q},\mathbb{P}}(Z)$. *Furthermore, this assumption combined to Proposition 3.2 leads to* $f_{X,\mathbb{Q}} = f_{X,\,g_Z^{\mathbb{Q},\mathbb{P}}}$. *This implies both that* $\mathbb{E}^{\tilde{\mathbb{Q}}}[X] = \mathbb{E}^{\mathbb{Q}}[X]$ *and* $\mathbb{Q}(F) = \tilde{\mathbb{Q}}(F)$ *for any event* $F \in \sigma(X)$, *the* σ*-algebra generated by* X. *However,* $\tilde{\mathbb{Q}}$ *and* $\mathbb{Q}$ *can still be distinct measures, i.e., it is possible to have* $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \neq \frac{d\mathbb{Q}}{d\mathbb{P}}$, *see Appendix B for a simple example. Indeed,* $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}$ *only ensures the distribution of the terminal value of the risk* X *is distorted in way to recover a price consistent with arbitrage-free pricing. Conversely, if* $\mathbb{Q}$ *is a martingale measure,* $\frac{d\mathbb{Q}}{d\mathbb{P}}$ *must alter the whole evolution of the risk dynamics through time, not only its terminal value distribution.*

The representation (3.5) of the distorted density of X provides useful information which is not directly given by the usual risk-neutral pricing approach. Indeed the weighting factor $\xi_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(\bar{F}_{X,\mathbb{P}}(x))\big)$ directly relates the physical density $f_{X,\mathbb{P}}(x)$ to the risk-neutral density $f_{X,\,g_Z^{\mathbb{Q},\mathbb{P}}}(x)$ for each possible value x of the risk X, allowing to understand how each value of X is weighted during the pricing. Conversely, since there is not necessarily a bijective relationship between the value of X and $\frac{d\mathbb{Q}}{d\mathbb{P}}$, it might not be straightforward to obtain $\frac{f_{X,\,g_Z^{\mathbb{Q},\mathbb{P}}}(x)}{f_{X,\mathbb{P}}(x)}$ directly from the expression of $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

3.4 Connections between the Wang transform and other pricing frameworks

There is a literature studying the connections between the Wang transform, the Esscher-Girsanov change of measure and the Bühlmann general equilibrium model. We now show that these results can be recovered by virtue of Theorem 3.1. We refer to Wang (2003) and Labuschagne and Offwood (2010) for the original proofs.[5]

First, let's see how the Wang transform can allow recovering prices provided by the Esscher-Girsanov change of measure. Suppose that Z is a $N(0,1)$ random variable under $\mathbb{P}$, and that its distribution is shifted to a $N(\theta,1)$ under $\mathbb{Q}$. As stated in Remark 3.1, the distortion operator allowing for such a change of distribution is the Wang transform $g_Z^{\mathbb{Q},\mathbb{P}}(u) = \Phi\big(\Phi^{-1}(u)+\theta\big)$. Moreover, one can readily show that the likelihood ratio is $\xi_Z^{\mathbb{Q},\mathbb{P}}(x) = e^{\theta x-\theta^2/2}$. Therefore, applying Theorem 3.1 with $\bar{F}_{Z,\mathbb{P}}(x) = 1-\Phi(x)$ gives

$$f_{X,g_Z^{\mathbb{Q},\mathbb{P}}}(x) = e^{\theta\Phi^{-1}(F_{X,\mathbb{P}}(x))-\theta^2/2} f_{X,\mathbb{P}}(x), \tag{3.6}$$

where $F_{X,\mathbb{P}}(x) \equiv 1-\bar{F}_{X,\mathbb{P}}(x)$. When $\theta = h\nu$ as defined in Labuschagne and Offwood (2010), we recuperate the Esscher-Girsanov change of measure as defined in Definition 3.1 of the latter paper.

Next, we illustrate how the Wang transform can allow recovering prices provided by the Bühlmann (1980) pricing equilibrium. The key assumption in Wang (2003) is that X and Z are co-monotone in the sense that they can be expressed as $X = F_{X,\mathbb{P}}^{-1}(U)$ and $Z = \Phi^{-1}(U)$, for some U that is uniformly distributed between 0 and 1. Under this assumption, we can use the expression $Z = \Phi^{-1}(F_{X,\mathbb{P}}(X))$ in (3.6) to obtain

$$f_{X,g_Z^{\mathbb{Q},\mathbb{P}}}(x) = \mathbb{E}^{\mathbb{P}}\Big[e^{\theta Z-\theta^2/2}\Big|X=x\Big] f_{X,\mathbb{P}}(x) = \frac{\mathbb{E}^{\mathbb{P}}\big[e^{\theta Z}\big|X=x\big]}{\mathbb{E}^{\mathbb{P}}\big[e^{\theta Z}\big]} f_{X,\mathbb{P}}(x). \tag{3.7}$$

which is the pricing (or risk-neutralized) distribution described in Bühlmann (1980).

The next two sections present several particular cases of distortions that are generated through the general construction approach of (3.1). Symmetries arising from the use of a survival function

[5] See Kijima (2006), Wang (2007), and Kijima and Muromachi (2008) for closely related work.

$\bar{F}_{Z,\mathbb{Q}}$ and inverse survival function $\bar{F}^{-1}_{Z,\mathbb{P}}$ that are within the same family of distributions are exploited; such symmetries lead to simplifications in the expression of the distortion function, reducing the number of parameters and thus making the distortions more convenient to work with.

4 Distortion operators for financial models

Next, we derive distortion operators compatible with arbitrage-free pricing under non-Gaussian extensions of the Black-Scholes model using results from the previous section. Empirically, it turns out that such extensions allow reproducing well-documented stylized facts of stock returns and options prices, such as the "volatility smiles" observed in option markets and skewed fat-tails distributions for stock returns.

Consider a continuous-time market, with time $t \in [0, T]$, containing a liquid risk-free asset growing at the risk-free rate r and a (possibly non-traded) underlying risky process $\{S_t\}_{t\in[0,T]}$ defined on a probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual assumptions. In Section 4.1, we model the underlying asset with an infinite activity Lévy process, and in Section 4.2, with a jump-diffusion process. Both approaches, now standard in the literature, have been widely applied in finance (see, e.g., Schoutens, 2003; Cont and Tankov, 2004).

4.1 Normal Inverse Gaussian distortion based on the Esscher martingale measure

Godin et al. (2012) propose a distortion operator that can recuperate the arbitrage-free prices for a non-Gaussian extension of the Black-Scholes model based on the Normal Inverse Gaussian (NIG) distribution and the mean-correcting martingale measure. Here, we show how their operator can be recovered as a particular case of the distortion operator (3.1). Moreover, to make this exercise rewarding, we extend the work of Godin et al. (2012) to the Esscher martingale measure, which provides a new NIG-based distortion operator that benefits from the very same advantages.

4.1.1 Dynamics under the physical measure

Under the physical measure, the underlying asset price follows an exponential NIG Lévy process with parameters $(\alpha, \beta, \delta, \mu)$. Its terminal value takes the form (see, e.g., Schoutens, 2003)

$$S_T = S_0 e^{X_T}, \qquad X_T \sim \text{NIG}(\alpha, \beta, \delta T, \mu T), \tag{4.1}$$

where $\text{NIG}(\alpha, \beta, \delta, \mu)$ is the Normal Inverse Gaussian distribution whose density is given by

$$\mathbf{nig}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha\delta K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\pi\sqrt{\delta^2 + (x-\mu)^2}} e^{\gamma\delta + \beta(x-\mu)},$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$, $\alpha > 0$, $\beta \in (-\alpha, \alpha)$, $\delta > 0$, $\mu \in \mathbb{R}$ and K_ν is the modified Bessel function of the second kind:

$$K_\nu(y) = \frac{1}{2}\int_0^\infty u^{\nu-1} e^{-\frac{1}{2}y(u+u^{-1})} du, \quad y > 0.$$

The distribution is a generalization of the normal distribution that allows for skewness and excess kurtosis. The cumulative distribution function and survival function of the $\text{NIG}(\alpha, \beta, \delta, \mu)$ are respectively denoted by $\mathbf{NIG}(x; \alpha, \beta, \delta, \mu)$ and $\overline{\mathbf{NIG}}(x; \alpha, \beta, \delta, \mu)$, $x \in \mathbb{R}$. Some useful remarks about this distribution are stated below.

Remark 4.1. If $X \sim \text{NIG}(\alpha, \beta, \delta, \mu)$ we have that $Y = aX + b$, for $a > 0$ and $b \in \mathbb{R}$, is such that $Y \sim \text{NIG}(\alpha/a, \beta/a, a\delta, a\mu + b)$.

Remark 4.2. The NIG distribution possesses the following symmetry property:

$$\overline{\mathbf{NIG}}(x; \alpha, \beta, \delta, 0) = \mathbf{NIG}(-x; \alpha, -\beta, \delta, 0).$$

4.1.2 Dynamics under the Esscher risk-neutral measure

It turns out that the market described above is incomplete, and therefore there exists an infinite number of risk-neutral measures (Eberlein and Jacod, 1997). We choose the Esscher martingale measure since it is a popular choice (e.g., Gerber and Shiu, 1994). Under this measure, the

process S is an exponential NIG Lévy process with parameters $(\alpha, \beta+\theta, \delta, \mu)$.[6] Under the Esscher martingale measure $\mathbb{Q}$, the terminal value of the underlying is such that

$$X_T \sim \text{NIG}\big(\alpha, \beta+\theta, \delta T, \mu T\big). \tag{4.2}$$

4.1.3 Derivation of the distortion operator

Using Remark 4.1, we can express (4.1) and (4.2) more compactly as

$$S_T = h(Z) \equiv S_0 \exp\Big\{\mu T + \sqrt{\delta/\alpha}\, Z\Big\}, \tag{4.3}$$

where the random variable Z is such that

$$Z \sim \begin{cases} \text{NIG}\big(\sqrt{\alpha\delta}, \beta\sqrt{\delta/\alpha}, T\sqrt{\alpha\delta}, 0\big), & \text{under } \mathbb{P}, \\ \text{NIG}\big(\sqrt{\alpha\delta}, (\beta+\theta)\sqrt{\delta/\alpha}, T\sqrt{\alpha\delta}, 0\big), & \text{under } \mathbb{Q}. \end{cases} \tag{4.4}$$

To obtain the expression of the distortion operator, we start from Definition 3.2 and simplify it using Proposition 3.1 and the symmetry property of Remark 4.2:

$$g_{S_T}^{\mathbb{Q},\mathbb{P}}(u) = g_{h(Z)}^{\mathbb{Q},\mathbb{P}}(u) = g_Z^{\mathbb{Q},\mathbb{P}}(u) = \Phi_{\mathbf{NIG}}^{\mathbb{Q}}\big(\Phi_{\mathbf{NIG}}^{-1}(u)\big), \tag{4.5}$$

where the following definitions are used:

$$\Phi_{\mathbf{NIG}}(x) \equiv \mathbf{NIG}\big(x; \xi, \zeta, T\xi, 0\big), \qquad \Phi_{\mathbf{NIG}}^{\mathbb{Q}}(x) \equiv \mathbf{NIG}\big(x; \xi, \zeta^{\mathbb{Q}}, T\xi, 0\big),$$

with $\xi \equiv \sqrt{\alpha\delta}$, $\zeta \equiv -\beta\sqrt{\delta/\alpha}$, $\zeta^{\mathbb{Q}} \equiv -(\beta+\theta)\sqrt{\delta/\alpha}$, and where $\Phi_{\mathbf{NIG}}^{-1}$ is the inverse of $\Phi_{\mathbf{NIG}}$. Note that due to symmetry property of Remark 4.2, the distortion (4.5) depends on only the three parameters $\xi, \zeta, \zeta^{\mathbb{Q}}$ instead of the five parameters $\alpha, \beta, \delta, \mu, \theta$ involved in the NIG distribution. Figure 1 presents the distorted survival function of a uniform $[0,1]$ distribution, i.e., the function

[6] Moreover, if S is the price process of a traded asset, then θ must be determined so that the discounted price process $\{S_t e^{-rt}\}_{t\in[0,T]}$ is a martingale under $\mathbb{Q}$ (see Schoutens, 2003, p. 79).

$u \to g_Z^{\mathbb{Q},\mathbb{P}}(1-u)$, for various distortions and sets of parameters. The upper left panel shows the NIG Esscher distortion (4.5). It can be seen that if $\zeta^{\mathbb{Q}}$ is greater (smaller) than ζ, some mass of the distribution is shifted to the right (left). $\zeta = \zeta^{\mathbb{Q}}$ leads to the identity distortion. Furthermore, increasing ξ dampens the effect of the disparity between ζ and $\zeta^{\mathbb{Q}}$. Finally, different sets of parameters can lead to very similar distortions (see the blue dotted and the orange dotted-dashed curves).

By Corollary 3.1, it follows that, up to discounting, the distortion (4.5) recovers the time-0 arbitrage-free price under the Esscher martingale measure $\mathbb{Q}$ of any $\mathbb{Q}$-integrable contingent claim $h(S_T)$ such that h is continuous and increasing. In fact, Black-Scholes-type formulas can be recuperated through (3.3) in a similar fashion as in Godin et al. (2012). Note that the difference between the NIG distortion from (4.5) and the one proposed by Godin et al. (2012) lies in the choice of the equivalent martingale measure.[7] The new Esscher-based NIG distortion from (4.5) exhibits the same improvements over the Wang transform than the one from Godin et al. (2012), e.g., it offer a greater variety of shapes that are suitable for pricing options under non-Gaussian conditions.

4.2 A distortion operator based on Kou (2002)'s jump-diffusion model

Consider now Kou (2002)'s jump-diffusion option pricing model. This model is based on the equilibrium framework of Section 2.1.1, where the utility function of the representative agent is assumed to be of the form $U(t,c) = e^{-\kappa t}\frac{c^\alpha}{\alpha}$ for $\alpha \in (0,1]$, and $U(t,c) = e^{-\kappa t}\ln c$ for $\alpha = 0$, with κ being a subjective deterministic discount factor. Kou's model offers an attractive tradeoff between realism and tractability. It is able to reproduce the leptokurtic feature of the return distribution and the "volatility smile" observed in option markets, yet it is simple enough to produce analytical formulas for call/put options, interest rate derivatives, and a variety of path-dependent options. We first describe Kou's model, and then derive the distortion operator allowing recovering some

[7] Godin et al. (2012) choose the mean-correcting martingale measure rather than the Esscher martingale measure. Under this measure, S is an exponential NIG Lévy process with parameters $(\alpha, \beta, \delta, \mu + \theta)$. One can verify that in this case the operator from Definition 3.2 indeed is the NIG distortion proposed in their Definition 2: $\Phi_{\mathbf{NIG}}(\Phi_{\mathbf{NIG}}^{-1}(u) + \theta T\sqrt{\alpha/\delta})$.

arbitrage-free option prices associated with this model.

As in Equation (7) of Kou (2002), the endowment process is modelled by the following jump-diffusion process under $\mathbb{P}$:

$$\frac{d\delta_t}{\delta_{t^-}} = \mu_1 dt + \sigma_1 dW_t^{(1)} + d\left[\sum_{i=1}^{N_t}(V_i - 1)\right], \tag{4.6}$$

where $\mu_1 \in \mathbb{R}$ and $\sigma_1 > 0$ are constants, $\left\{W_t^{(1)}\right\}_{t\in[0,T]}$ is a Wiener process, $\{N_t\}_{t\in[0,T]}$ is a standard Poisson process with intensity $\lambda > 0$, $\{V_i\}_{i\geq 1}$ is a sequence of i.i.d. non-negative random variables, and all sources of randomness are independent.

As in Equation (13) of Kou (2002), the underlying follows the jump-diffusion process

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma dW_t + d\left[\sum_{i=1}^{N_t}\left(V_i^{\beta} - 1\right)\right], \tag{4.7}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, and $\beta \in \mathbb{R}$ are constants. The new Wiener process $\left\{W_t\right\}_{t\in[0,T]}$ has a constant correlation $\rho \in [-1, 1]$ to $W^{(1)}$ and is independent from the other sources of randomness.

4.2.1 Dynamics under the physical measure

The distribution of the log-size jumps is modelled by an Asymmetric Double Exponential distribution $\text{ADE}(\eta_1, \eta_2, p)$. The density of this distribution is

$$\mathbf{ade}(x; \eta_1, \eta_2, p) = p\eta_1 e^{-\eta_1 x}\,\mathbb{1}_{x\geq 0} + (1-p)\eta_2 e^{\eta_2 x}\,\mathbb{1}_{x<0}, \qquad x \in \mathbb{R}. \tag{4.8}$$

The parameter domain is $\eta_1 > 0$, $\eta_2 > 0$, $p \in [0, 1]$. As stated in Remark 4.3, the ADE distribution is closed under scaling.

Remark 4.3. If $X \sim \text{ADE}(\eta_1, \eta_2, p)$, then $Y \equiv aX$ with $a \neq 0$ is such that

$$Y \sim \begin{cases} \text{ADE}(\eta_1/a, \eta_2/a, p), & a > 0, \\ \text{ADE}(\eta_2/|a|, \eta_1/|a|, 1-p), & a < 0. \end{cases}$$

Solving the stochastic differential equation (4.7) yields

$$S_T = S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T + \beta \sum_{i=1}^{N_T} Y_i \right\}, \tag{4.9}$$

with $\{Y_i \equiv \ln V_i\}_{i\geq 1} \overset{\text{i.i.d.}}{\sim} \text{ADE}(\eta_1, \eta_2, p)$, $W_T \sim N(0,T)$, and $N_T \sim \text{Poisson}(\lambda T)$.

4.2.2 Dynamics under the risk-neutral measure

Theorem 1 of Kou (2002) describes the dynamics under the risk-neutral measure $\mathbb{Q}$ stemming from the Lucas (1978)-type equilibrium framework and the conditions for the existence of this measure. From this theorem, it can be shown that under $\mathbb{Q}$ the terminal value of the underlying is such that

$$S_T = S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2} - \rho\sigma\sigma_1(1-\alpha)\right)T + \sigma W_T^{\mathbb{Q}} + \beta \sum_{i=1}^{N_T} Y_i \right\}, \tag{4.10}$$

with $\{Y_i\}_{i\geq 1} \overset{\text{i.i.d.}}{\sim} \text{ADE}\big(\eta_1^{\mathbb{Q}}, \eta_2^{\mathbb{Q}}, p^{\mathbb{Q}}\big)$, $W_T^{\mathbb{Q}} \sim N(0,T)$, and $N_T \sim \text{Poisson}(\lambda^{\mathbb{Q}} T)$, where

$$\eta_1^{\mathbb{Q}} = \eta_1 - \alpha + 1, \qquad \eta_2^{\mathbb{Q}} = \eta_2 + \alpha - 1, \qquad p^{\mathbb{Q}} = \frac{p}{\zeta}\frac{\eta_1}{\eta_1 - \alpha + 1}, \qquad \lambda^{\mathbb{Q}} = \zeta\lambda, \tag{4.11}$$

with $\zeta \equiv \frac{p\eta_1}{\eta_1 - \alpha + 1} + \frac{(1-p)\eta_2}{\eta_2 + \alpha - 1}$. The three sources of risk $\{Y_i\}_{i\geq 1}$, $W_T^{\mathbb{Q}}$ and N_T are still independent under $\mathbb{Q}$.

4.2.3 Derivation of the distortion operator

The survival function defined below will be useful in defining the distortion operator. We refer to Appendix B of Kou (2002) for results that ease its numerical implementation.

Definition 4.1. Let the following random variables be independent: $\mathcal{Z} \sim N(\mu, \sigma^2)$ is a normal variable, $P \sim \text{Poisson}(\lambda)$ is a Poisson variable, and $\{Y_i\}_{i\geq 1} \overset{\text{i.i.d.}}{\sim} \text{ADE}(\eta_1, \eta_2, p)$. The cumulative distribution function of the *Normal Compound Poisson distribution* $NCP(\lambda, \eta_1, \eta_2, p, \sigma, \mu)$ is defined as

$$\mathbf{NCP}\big(x; \lambda, \eta_1, \eta_2, p, \sigma, \mu\big) \equiv \Pr\left(\mathcal{Z} + \sum_{i=1}^{P} Y_i \leq x \right), \qquad x \in \mathbb{R}. \tag{4.12}$$

The survival function is denoted by $\overline{\mathbf{NCP}}\big(x;\lambda,\eta_1,\eta_2,p,\sigma,\mu\big) = 1 - \mathbf{NCP}\big(x;\lambda,\eta_1,\eta_2,p,\sigma,\mu\big)$.

Using Remark 4.3, it is a straightforward exercise to show that the NCP distribution is closed under affine transformations, as stated below in Remark 4.4. Moreover, the cumulative distribution function and the survival function of the NCP distribution are related by the symmetry property stated in Remark 4.5.

Remark 4.4. If $X \sim \mathrm{NCP}(\lambda,\eta_1,\eta_2,p,\sigma,\mu)$ then $Y = aX + b$ with $a \neq 0$ is such that

$$Y \sim \begin{cases} \mathrm{NCP}\big(\lambda,\eta_1/a,\eta_2/a,p,a\sigma,a\mu+b\big), & a > 0, \\ \mathrm{NCP}\big(\lambda,\eta_2/|a|,\eta_1/|a|,1-p,|a|\sigma,a\mu+b\big), & a < 0. \end{cases}$$

Remark 4.5. The NCP distribution possesses the following symmetry:

$$\overline{\mathbf{NCP}}(x;\lambda,\eta_1,\eta_2,p,\sigma,\mu) = \mathbf{NCP}(-x;\lambda,\eta_2,\eta_1,1-p,\sigma,-\mu).$$

We can make use of Remark 4.4 to express (4.9) and (4.10) more compactly as

$$S_T = h(Z) \equiv S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right\}, \tag{4.13}$$

where the random variable Z is such that

$$Z \sim \begin{cases} \mathrm{NCP}\big(\lambda T,\tilde{\eta}_1,\tilde{\eta}_2,\tilde{p},1,0\big), & \text{under } \mathbb{P}, \\ \mathrm{NCP}\big(\lambda^{\mathbb{Q}}T,\tilde{\eta}_1^{\mathbb{Q}},\tilde{\eta}_2^{\mathbb{Q}},\tilde{p}^{\mathbb{Q}},1,-\rho\sigma_1(1-\alpha)\sqrt{T}\big), & \text{under } \mathbb{Q}, \end{cases} \tag{4.14}$$

where the tilde parameters are defined in terms of

$$\nu \equiv \sigma\sqrt{T}/|\beta| \tag{4.15}$$

as follows:

$$\begin{aligned}
&(\text{if } \beta \geq 0) \quad \tilde{\eta}_1 = \eta_1 \nu, \quad \tilde{\eta}_1^{\mathbb{Q}} = \eta_1^{\mathbb{Q}} \nu, \quad \tilde{\eta}_2 = \eta_2 \nu, \quad \tilde{\eta}_2^{\mathbb{Q}} = \eta_2^{\mathbb{Q}} \nu, \quad \tilde{p} = p, \quad \tilde{p}^{\mathbb{Q}} = p^{\mathbb{Q}}, \\
&\quad \text{or} \\
&(\text{if } \beta < 0) \quad \tilde{\eta}_1 = \eta_2 \nu, \quad \tilde{\eta}_1^{\mathbb{Q}} = \eta_2^{\mathbb{Q}} \nu, \quad \tilde{\eta}_2 = \eta_1 \nu, \quad \tilde{\eta}_2^{\mathbb{Q}} = \eta_1^{\mathbb{Q}} \nu, \quad \tilde{p} = 1 - p, \quad \tilde{p}^{\mathbb{Q}} = 1 - p^{\mathbb{Q}}.
\end{aligned} \tag{4.16}$$

To obtain the expression of the distortion operator, we start from Definition 3.2 and simplify it by using Proposition 3.1 and the symmetry property of Remark 4.5:

$$g_{S_T}^{\mathbb{Q},\mathbb{P}}(u) = g_{h(Z)}^{\mathbb{Q},\mathbb{P}}(u) = g_Z^{\mathbb{Q},\mathbb{P}}(u) = \Phi_{\mathbf{NCP}}^{\mathbb{Q}}\Big(\Phi_{\mathbf{NCP}}^{-1}(u) - \rho\sigma_1(1-\alpha)\sqrt{T}\Big), \tag{4.17}$$

where the following definitions are used:

$$\Phi_{\mathbf{NCP}}(x) \equiv \mathbf{NCP}\big(x; \lambda T, \tilde{\eta}_2, \tilde{\eta}_1, 1-\tilde{p}, 1, 0\big), \qquad \Phi_{\mathbf{NCP}}^{\mathbb{Q}}(x) \equiv \mathbf{NCP}\big(x; \lambda^{\mathbb{Q}} T, \tilde{\eta}_2^{\mathbb{Q}}, \tilde{\eta}_1^{\mathbb{Q}}, 1-\tilde{p}^{\mathbb{Q}}, 1, 0\big),$$

with $\Phi_{\mathbf{NCP}}^{-1}$ defined as the inverse of $\Phi_{\mathbf{NCP}}$. Thus, due to the symmetry outlined in Remark 4.5, as indicated for instance in Section 6.2, the family of distortions obtained through (4.17) can be spanned by only a subset of possible parameters of the jump-diffusion model, which is a convenient simplification.

From Corollary 3.1, it follows that the distortion operator (4.17) recovers prices generated by the equilibrium setup described above. For instance, through (3.3), the distortion can recuperate the European call formula (20) from Kou (2002). This distortion operator produces a risk-neutralized distribution that can be viewed as capturing premiums for both jumps and Brownian motion risks.

5 Distortion operators for insurance models

We now derive distortion operators for two insurance pricing models. The first is based on the size-biased pricing principle for the generalized beta of the second kind distribution, and the second is based on the Esscher principle for the gamma distribution. In both cases, the underlying

risk is an insurance claim represented by a positive random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

We adopt the conventions used in Klugman et al. (2012) for the incomplete beta function and the incomplete gamma function. The incomplete beta function is defined as

$$\beta(x;\tau,\alpha) \equiv \frac{\Gamma(\tau+\alpha)}{\Gamma(\tau)\Gamma(\alpha)} \int_0^x t^{\tau-1}(1-t)^{\alpha-1}dt, \qquad \tau>0, \quad \alpha>0, \quad x\in(0,1), \tag{5.1}$$

and the (lower) incomplete gamma function is defined as

$$\Gamma(x;\alpha) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1}e^{-t}dt, \qquad \alpha>0, \quad x>0, \tag{5.2}$$

where $\Gamma(\alpha) \equiv \int_0^\infty t^{\alpha-1}e^{-t}dt$ is the gamma function.

5.1 A distortion based on the generalized beta of the second kind distribution

The generalized beta of the second kind (GB2) distribution, sometimes called the transformed beta distribution, is a member of the celebrated Pearson system and was first proposed as a model of the size-of-loss distribution in actuarial sciences by Venter (1983). This large family of heavy-tailed distributions contains the Burr, generalized Pareto, generalized gamma, log-t, and other commonly used distributions. It provides a fairly flexible form that can be used to model highly skewed loss distributions such as those typically observed in non-life insurance (Cummins et al., 1990). Here, we use the GB2 to model the risk X. A thorough account of other size-of-loss distributions can be found in Kleiber and Kotz (2003) and in Klugman et al. (2012).

The GB2 distribution has heavy tails, and therefore only a few of the moments exist (Kleiber, 1997). This implies that some risk-neutral measures, such as the Esscher measure, may exist only for a certain range of the shape parameters. Among the possible risk-neutral measures, the size-biased subclass is an interesting choice as it preserves the shape of the GB2 family, yielding a simple Wang-like distortion operator. We underline that our approach also applies to non shape-preserving changes of measure, although it may require tedious algebra to simplify the form of the distortion function.

5.1.1 *Distribution under the physical measure*

Under the physical measure $\mathbb{P}$, we assume the risk X follows a GB2$(\alpha, \theta, \gamma, \tau)$ distribution as defined in Klugman et al. (2012). The density of this distribution is

$$\mathbf{gb2}(x; \alpha, \theta, \gamma, \tau) = \frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)} \frac{\gamma(x/\theta)^{\gamma\tau}}{x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}, \qquad x \geq 0, \tag{5.3}$$

and its cumulative distribution function is

$$\mathbf{GB2}(x; \alpha, \theta, \gamma, \tau) = \beta\left(\frac{(x/\theta)^{\gamma}}{1+(x/\theta)^{\gamma}}; \tau, \alpha\right), \qquad x \geq 0. \tag{5.4}$$

The parameter domain of this distribution is $\alpha > 0$, $\theta > 0$, $\gamma > 0$, $\tau > 0$. The parameter controlling the scale is θ. The other parameters control the tail behaviour and the shape in general. It is interesting to note that the log-normal distribution is a limiting case of the GB2 distribution, as stated in the following remark.

Remark 5.1 (Log-normal limit)**.** A log-normal distribution with density $\mathbf{logN}(x; \mu, \sigma) \equiv \frac{\phi\left(\frac{\ln x - \mu}{\sigma}\right)}{\sigma x}$, $x \geq 0$, is obtained as a limiting case of the GB2$(\alpha, \theta, \gamma, \tau)$ distribution when $\alpha \to \infty$, $\gamma \to 0$, $\theta = (\alpha\gamma^2\sigma^2)^{1/\gamma}$ and $\tau = (\gamma\mu + 1)/(\sigma^2\gamma^2)$ (from McDonald, 1987).

The survival function under $\mathbb{P}$ of the risk X is thus given by

$$\bar{F}_{X,\mathbb{P}}(x) \equiv \mathbb{P}(X > x) = 1 - \beta\left(\frac{(x/\theta)^{\gamma}}{1+(x/\theta)^{\gamma}}; \tau, \alpha\right), \qquad x \geq 0. \tag{5.5}$$

5.1.2 *Distribution under the risk-neutral measure*

Consider any constant a such that $-\gamma\tau < a < \alpha\gamma$. Under the pricing measure $\mathbb{Q}_X$ induced by the size-biased premium calculation principle,[8] the density of X is given by

$$\frac{x^a}{\mathbb{E}^{\mathbb{P}}[X^a]} \mathbf{gb2}(x; \alpha, \theta, \gamma, \tau) \propto \mathbf{gb2}(x; \tilde{\alpha}, \theta, \gamma, \tilde{\tau}), \tag{5.6}$$

[8] Here the pricing measure $\mathbb{Q}_X$ depends on the risk X. Indeed, in the pricing approach of Furman and Zitikis (2008) described in Section 2.2, the pricing kernel depends on the risk being priced, which explains why the measure $\mathbb{Q}_X$ is made dependent on X.

where

$$\tilde{\alpha} \equiv \alpha - a/\gamma, \qquad \tilde{\tau} \equiv \tau + a/\gamma. \tag{5.7}$$

By normalization, the right-hand side of (5.6) implies that $X \sim \text{GB2}(\tilde{\alpha}, \theta, \gamma, \tilde{\tau})$ under $\mathbb{Q}_X$.

The survival function of X under $\mathbb{Q}_X$ is therefore given by

$$\bar{F}_{X,\mathbb{Q}_X}(x) \equiv \mathbb{Q}_X(X > x) = 1 - \beta\left(\frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}; \tilde{\tau}, \tilde{\alpha}\right), \qquad x \geq 0. \tag{5.8}$$

5.1.3 Derivation of the distortion operator

To obtain the distortion operator, the first step is to use (5.8) in Definition 3.2 to obtain

$$g_X^{\mathbb{Q}_X,\mathbb{P}}(u) \equiv \bar{F}_{X,\mathbb{Q}_X}\left(\bar{F}_{X,\mathbb{P}}^{-1}(u)\right) = 1 - \beta\left(\frac{\left(\bar{F}_{X,\mathbb{P}}^{-1}(u)/\theta\right)^\gamma}{1+\left(\bar{F}_{X,\mathbb{P}}^{-1}(u)/\theta\right)^\gamma}; \tilde{\tau}, \tilde{\alpha}\right). \tag{5.9}$$

Next, we use $x = \bar{F}_{X,\mathbb{P}}^{-1}(u)$ in (5.5) to obtain the following relation:

$$u = 1 - \beta\left(\frac{\left(\bar{F}_{X,\mathbb{P}}^{-1}(u)/\theta\right)^\gamma}{1+\left(\bar{F}_{X,\mathbb{P}}^{-1}(u)/\theta\right)^\gamma}; \tau, \alpha\right) \quad \Rightarrow \quad \frac{\left(\bar{F}_{X,\mathbb{P}}^{-1}(u)/\theta\right)^\gamma}{1+\left(\bar{F}_{X,\mathbb{P}}^{-1}(u)/\theta\right)^\gamma} = \beta^{-1}\big(1-u; \tau, \alpha\big),$$

where the expression of the right-hand side is obtained by rearranging the terms and taking the inverse incomplete beta function β^{-1}. Using this expression in (5.9) gives the GB2 distortion:

$$g_X^{\mathbb{Q}_X,\mathbb{P}}(u) = 1 - \beta\big(\beta^{-1}(1-u; \tau, \alpha); \tilde{\tau}, \tilde{\alpha}\big)\,. \tag{5.10}$$

Note that the distortion (5.10) does not depend on θ. The bottom left panel of Figure 1 presents the distorted survival function of a uniform $[0,1]$ distribution for the GB2 distortion (5.10). The identity distortion corresponds to $a/\gamma = 0$. It can be seen that either increasing a/γ, decreasing τ or decreasing α leads to some mass of the distribution being shifted to the right. Moreover, comparing the blue dotted curve to the orange dotted-dashed curve shows that simultaneously suitably reducing α and increasing τ can allow increasing the skewness of the distribution.

An interesting special case of the GB2 distortion is obtained for $\tau = \alpha = 1$, for which it is a

straightforward exercice to prove that we obtain $g_X^{\mathbb{Q}_X,\mathbb{P}}(u) = 1 - \beta\big(1-u; 1+a/\gamma, 1-a/\gamma\big)$. This distortion is a variant of the beta transform in Wirch and Hardy (1999). In the latter paper, the authors argue that the corresponding distortion risk measure has advantages over the expected shortfall because it utilizes the whole distribution rather than focusing only the tail.

An even more interesting property of the GB2 distortion is that it reduces to the Wang transform $g_X^{\mathbb{Q}_X,\mathbb{P}}(u) = \Phi\big(\Phi^{-1}(u) + a\sigma\big)$ in the log-normal limiting case stated in Remark 5.1.[9] The GB2 distortion can therefore be seen as a flexible generalization of the Wang transform that can account for heavier tails. It is a well-known fact that correctly modelling the tail behaviour is crucially important for premium calculation and risk measurement. Indeed, the Wang transform must be modified to capture the heavy tail feature since this transform is meant to be applied in the context of Gaussian risks.

5.2 An extension of the proportional hazards transform

We now turn to the Esscher pricing principle. Because this principle relies on an exponential weight function, the risk-neutral measure does not exist for some of the heavy-tailed distributions in the GB2 family. It is however applicable to the gamma subfamily, which includes the exponential and chi-squared distributions. Here, we suppose that the insurance claim X is gamma distributed, and we derive the distortion that is consistent with the Esscher principle. Interestingly, it turns out that the proportional hazards (PH) transform (Wang, 1995) is a special case of this distortion.

5.2.1 Distribution under the physical measure

Under the physical measure $\mathbb{P}$, we assume the risk X follows a Gamma(α, θ) distribution. The parameter domain of this distribution is $\alpha > 0$, $\theta > 0$. The density and the cumulative distribution function are given by

$$\mathbf{gamma}(x;\alpha,\theta) = \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)}, \qquad \mathbf{Gamma}(x;\alpha,\theta) = \Gamma\big(x/\theta;\alpha\big), \qquad x \geq 0. \tag{5.11}$$

[9] Let X be a log-normal variable distributed with density $\mathbf{logN}(x;\mu,\sigma) \equiv \phi\big(\frac{\ln x-\mu}{\sigma}\big)/(\sigma x)$, $x > 0$. It can be shown that $\frac{x^a}{\mathbb{E}^{\mathbb{P}}[X^a]}\mathbf{logN}(x;\mu,\sigma) = \mathbf{logN}(x;\mu+a\sigma^2,\sigma)$. The distortion operator, obtained in a similar fashion as in Section 5.1.3, turns out to be the Wang transform $g_X^{\mathbb{Q},\mathbb{P}}(u) = \Phi\big(\Phi^{-1}(u) + a\sigma\big)$.

The survival function of X under $\mathbb{P}$ is thus $\bar{F}_{X,\mathbb{P}}(x) = 1 - \Gamma\big(x/\theta;\alpha\big)$, $x \geq 0$.

5.2.2 Distribution under the risk-neutral measure

Provided $\mathbb{E}^{\mathbb{P}}[e^{aX}] < \infty$, the density of X under its associated pricing measure $\mathbb{Q}_X$ induced by the Esscher transform is given by

$$\frac{e^{ax}}{\mathbb{E}^{\mathbb{P}}[e^{aX}]}\mathbf{gamma}(x;\alpha,\theta) \propto \mathbf{gamma}(x;\alpha,\tilde{\theta}), \qquad \tilde{\theta} \equiv \frac{\theta}{1-a\theta}. \tag{5.12}$$

By normalization, the right-hand side indeed implies that $X \sim \text{Gamma}(\alpha,\tilde{\theta})$ under $\mathbb{Q}_X$. Note that $a < 1/\theta$ is required to ensure the existence of this measure.

The survival function of X under $\mathbb{Q}_X$ is thus $\bar{F}_{X,\mathbb{Q}_X}(x) = 1 - \Gamma\big(x/\tilde{\theta};\alpha\big)$, $x \geq 0$.

5.2.3 Derivation of the distortion operator

The derivation of the distortion operator follows the same steps as in Section 5.1.3. We obtain

$$g_X^{\mathbb{Q}_X,\mathbb{P}}(u) = 1 - \Gamma\big(\Gamma^{-1}(1-u;\alpha)[1-a\theta];\alpha\big), \tag{5.13}$$

where Γ^{-1} is the inverse of the incomplete gamma function Γ given in (5.2). Note that this distortion depends on only two parameters α and $a\theta$, despite the fact that three parameters α, θ, a are required to fully characterize both the physical and risk-neutral distributions of the risk. The bottom right panel of Figure 1 presents the distorted survival function of a uniform $[0,1]$ distribution for the Gamma Esscher distortion (5.13). The identity distortion corresponds to $a\theta = 0$. A strictly positive (negative) value of $a\theta$ shifts some mass of the distribution toward the right (left). Increasing the value of α amplifies this effect. Moreover, comparing the gray dashed curve to the dark green full curve shows that simultaneously reducing α and increasing $a\theta$ enables increasing the skewness of the distribution.

Note that for the special case $\alpha = 1$ (i.e., the exponential distribution) we obtain $g_X^{\mathbb{Q}_X,\mathbb{P}}(u) = u^{1-a\theta}$, which is the so-called PH transform (Wang, 1995). Our gamma-based distortion can therefore be seen as a natural generalization of the PH transform obtained from a more flexible distribution.

Moreover, our analysis further contributes to the discussions in Wang (1996) regarding the connections between risk-neutral valuation and the PH transform.

6 Empirical applications to CAT bonds

Catastrophe (CAT) bonds are insurance-linked securities devised by insurers and reinsurers to transfer natural disaster risks to the capital markets.[10] Several authors have proposed to price CAT bonds using the theory of contingent claim pricing for jump-diffusion processes (e.g., Vaugirard, 2003). These CAT bond pricing models usually involve extensive information that are not publicly disclosed (e.g., the trigger level or strike price).[11] It ensues that these types of models have yet to be tested empirically. In fact, most empirical studies on CAT bond spreads have instead relied on pure regression models (e.g., Bodoff and Gan, 2012; Braun, 2015). Meanwhile, explaining CAT bond spreads using distortion operators has been suggested by several works; the calibration of the Wang transform to CAT bond spreads is carried out, for instance, in Wang (2004) and in Galeotti et al. (2013). In this section, we employ our approach to perform further empirical tests on such models as it enables us to reformulate these in terms of the information available to the econometrician (e.g., the expected loss, the probability of first loss, the probability of last loss). Extending this stream of empirical studies, we make several interesting findings; in particular, it turns out that incorporating CAT risk and CAT risk-aversion into the distortion is important for explaining CAT bond spreads. This is established by calibrating a jump-diffusion distortion to market spreads. By way of a mixture of normal and exponential distributions, we also offer a simplified approximation to the latter distortion.

[10] CAT bond transactions usually involve a Special-Purpose Vehicle (SPV), located in a tax-efficient jurisdiction, that sells catastrophe protection to a ceding (re)insurer in the form of a reinsurance contract. The SPV then effectively transfers its risk exposure by issuing CAT bond tranches to capital market investors. In order to offer a virtually pure exposure to the natural disaster risk, the proceeds of the issuance are invested by the SPV in highly rated short-term assets that are held in a collateral account.

[11] Available databases for CAT bond transactions include the Artemis.bms deal directory, and the reports published by Aon Benfield, Swiss Re, Plenum, and Lane Financial LLC.

6.1 Setup

Let the payout to the ceding (re)insurer be contingent upon a risk X breaching a pre-agreed attachment point b, in which case the collateral is liquidated to reimburse the sponsor up to the par amount A paid by the investor at the issue date. If there is no triggering event during the term of the CAT bond, which is typically between one and four years, the principal is returned to the investor plus a coupon payment with spread S above the risk-free rate. Under the distortion premium principle, the spread is given by (from Galeotti et al., 2013)

$$\mathrm{S} = \frac{1}{A}\int_b^{b+A} g\big(\bar{F}_{X,\mathbb{P}}(x)\big)dx, \qquad \bar{F}_{X,\mathbb{P}}(x) \equiv \mathbb{P}(X > x). \tag{6.1}$$

The integral can be approximated using the trapezoidal rule:

$$\mathrm{S} \approx \big(g(\mathrm{PFL}) + g(\mathrm{PLL})\big)/2, \qquad \mathrm{PFL} \equiv \bar{F}_{X,\mathbb{P}}(b), \quad \mathrm{PLL} \equiv \bar{F}_{X,\mathbb{P}}(b+A). \tag{6.2}$$

If either of the probability of first loss PFL or the probability of last loss PLL is not available, the integral can be approximated using the rectangle method:

$$\mathrm{S} \approx g(\mathrm{EL}), \qquad \mathrm{EL} \equiv \frac{1}{A}\int_b^{b+A} \bar{F}_{X,\mathbb{P}}(x)dx. \tag{6.3}$$

The approximation (6.2) would be exact if $g \circ \bar{F}_{X,\mathbb{P}}$ was linear over $[b, b+A]$, whereas (6.3) would be exact if g was linear over $[\bar{F}_{X,\mathbb{P}}(b), \bar{F}_{X,\mathbb{P}}(b+A)]$. Such approximations could produce biases in calculations; for instance, if $g \circ \bar{F}_{X,\mathbb{P}}$ is convex over $[b, b+A]$ (a typical case since the latter function is decreasing and bounded from below), the approximation (6.2) would overshoot the spread. However, it is very difficult to determine the overall impact of such biases as the latter might be offset by the calibration of distortion parameters.

A data set containing 508 CAT bond tranches issued between 1997 and 2016 has been made available to us by Artemis.bm. Roughly half of the CAT bonds in the sample cover the United States, approximatively one quarter are multi-territory, and the rest cover Europe, Japan or other

areas. About half of the CAT tranches are multi-peril, and others are mainly wind-specific or earthquake-specific. For each of the 508 transactions, the available information includes the per annum spread S and expected loss EL. The probability of first loss PFL and the probability of last loss PLL are however available only for 284 transactions. Calibration is therefore carried out under the rectangle method (6.3), by minimizing the sum of squared pricing errors. Fitting adequacy is investigated using nonparametric regressions to detect systematic pricing errors; see Azzalini et al. (1989), Zheng (1996), and Li and Racine (2007) for works on nonparametric specification testing.

6.2 Pricing CAT bonds using a jump-diffusion distortion

The jump-diffusion distortion (4.17) is an interesting candidate for pricing CAT bonds because it can incorporate jump risk in a very straightforward way. Several frameworks based on jump-diffusion processes have in fact been proposed to price such insurance-linked securities (e.g., Vaugirard, 2003). In this section, instead of using the Wang transform as in Wang (2004) and Galeotti et al. (2013), we calibrate our jump-diffusion distortion to CAT bond spreads. Furthermore, we investigate whether jump risk is priced by the market.

Suppose the jumps are used to model natural disasters, i.e., the jump-diffusion process S in the model presented in Section 4.2 is used here to represent a catastrophe loss index whose dynamics is affected by positive jumps. Such events can only lower the aggregate endowment since the latter is a representation of the aggregate production of wealth in an economy which cannot be positively impacted by a catastrophe-related insurance loss. Therefore, we are interested in the special case $p = 0$ and $\beta < 0$. Indeed, $p = 0$ ensures the endowment process incurs no positive jump, while $\beta < 0$ ensures negative jumps in the endowment process generate positive jumps in the loss index.

For this special case, it is a straightforward exercise to show that the distortion operator (4.17) can be simplified and rewritten as follows:

$$g_{\varphi,\lambda,\eta,\nu}(u) \equiv \Upsilon_{\lambda\nu,\frac{\eta}{\nu}}\big(\Upsilon^{-1}_{\lambda,\eta}(u) + \varphi\big), \tag{6.4}$$

where $\lambda > 0$, $\eta > 0$, $\nu \geq 1$, $\varphi \in \mathbb{R}$, and $\Upsilon_{\lambda,\eta}^{-1}$ is the inverse of the function $\Upsilon_{\lambda,\eta}$ defined below.[12]

Definition 6.1. Let $\{Y_i\}_{i\geq 1}$ be a sequence of i.i.d. exponential random variables with rate $\eta > 0$, P be a Poisson random variable with rate $\lambda > 0$, and $\mathcal{Z}$ be a standard normal random variable. Assume $\{Y_i\}_{i\geq 1}$, P and $\mathcal{Z}$ are independent. Define the following cumulative distribution function:

$$\Upsilon_{\lambda,\eta}(x) \equiv \Pr\left(\mathcal{Z} - \sum_{i=1}^{P} Y_i \leq x\right), \qquad x \in \mathbb{R}. \tag{6.5}$$

The top right panel of Figure 1 illustrates the distorted survival function of a uniform $[0, 1]$ distribution for the distortion (6.4). The identity distortion corresponds to $\varphi = 0$ and $\nu = 1$. Having either $\varphi > 0$ or $\nu > 1$ shifts some mass of the distribution toward the right. Increasing λ can either amplify (dashed gray curve versus dashed red curve) or dampen (dotted-dashed orange curve versus dotted blue curve) this effect. Moreover, comparing the dashed red curve to the full dark green curve shows that the jump-diffusion distortion is sufficiently flexible to generate various levels of skewness in the distorted distribution.

We calibrate the distortion depicted by (6.4) using our CAT bond tranches data set. It appears there is a large (potentially infinite) set of solutions that yield nearly identical mean squared errors. For example, the solution $(\hat{\varphi}, \hat{\lambda}, \hat{\eta}, \hat{\nu}) = (0.02, 0.20, 4.50, 2.92)$ provides virtually the same distortion than (0.11, 0.23, 2.00, 1.87). This is not surprising as it is well-known that distributions stemming from jump-diffusion models suffer from parameter identification issues as discussed for instance in Remark 6.6.1 of Rémillard (2016). However, this is not problematic in the current framework; two sets of parameters yielding distributions that are almost identical will lead to almost identical distortions. One should therefore be indifferent with respect to the choice between any of such distortions.

We are also interested in determining whether jump risk is priced by the market. We seek an answer to this research question by calibrating the distortion operator under the constraint $\nu = 1$, which states that the jump amplitude and frequency are unchanged under the risk-neutral

[12] Note that the parameters λ, η, ν, and φ are defined differently than in preceding sections; the latter definition corresponds to the simplified econometric specification.

measure, i.e., that the market does not exhibit jump risk aversion. The calibration results are presented in Figure 2, where we can see that assuming the absence of jump risk aversion leads to a mis-specified model as shown by the residuals which display systematic deviations from zero. This provides, for the first time, interesting evidence that jump-diffusion models are appropriate for pricing CAT bonds, but that investors are risk-averse to natural disasters.

6.3 Results for other distortions

One potential disadvantage of the jump-diffusion distortion is the use of sophisticated analytical results on the Hh special function from mathematical physics in order to compute it efficiently (see, e.g., Kou, 2002, Appendix B). Hence, it would be interesting to find a readily implementable distortion that also fits well to CAT bond spreads.

Results for calibrating the GB2 distortion (5.10) to CAT bond market spreads are exhibited in Figure 3. Graph 3(b) reveals systematic pricing errors, exacerbated at small values of the expected loss EL. Unreported tests available upon request from the authors show that other distortion operators, such as the two NIG distortions and Wang (2004)'s distortion based on the Student-t distribution, suffer from similar issues. Hence, such distortions are less suitable for explaining CAT bond spreads. This might appear surprising as the GB2 family is known for its flexibility in modeling insurance losses (Cummins et al., 1990). On the other hand, the theoretical literature on CAT bonds and other insurance-linked securities advocate jump-diffusion processes and compound Poisson processes to model catastrophe loss processes.[13] In the previous sections, we have presented empirical evidence supporting the latter models.

6.4 A simple class of distortion operators to price CAT bonds

This section offers a simple class of Esscher-type distortions based on a mixture of distributions to characterize catastrophe risks. It is shown such distortions can provide an accurate representation of CAT bond spreads while being straightforward to implement in practice.

[13] See Vaugirard (2003), Nowak and Romaniuk (2013), Ma and Ma (2013), Perrakis and Boloorforoosh (2013) and Lai et al. (2014) for a non-exhaustive list of such works.

6.4.1 General framework

Let X be a strictly positive absolutely continuous random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ representing a risk. We assume X has the following representation allowing to express its distribution as a probability mixture:

$$X = e^{\mu + \sigma Z_I}, \tag{6.6}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants, I is a discrete random variable such that

$$\mathbb{P}(I = i) = p_i, \qquad i \in \{1, \ldots, m\}, \qquad \sum_{i=1}^{m} p_i = 1, \tag{6.7}$$

and the $Z_1, \ldots, Z_m$ are independent random variables with densities denoted by $f_1, \ldots, f_m$. Hence,

$$\mathbb{P}(Z_I \leq z) = \sum_{i=1}^{m} p_i F_i(z), \qquad F_i(z) \equiv \int_{-\infty}^{z} f_i(x)dx, \qquad z \in \mathbb{R}, \tag{6.8}$$

as I is presumed independent from the $Z_1, \ldots, Z_m$. Note that expressing the risk in the form (6.6) is convenient as Proposition 3.1 entails that $g_X^{\mathbb{Q},\mathbb{P}} = g_{Z_I}^{\mathbb{Q},\mathbb{P}}$, i.e., the associated distortion does not depend on μ nor σ, and thus Z_I can be shifted and scaled conveniently as to reduce the number of parameters of the distortion by two (see the example in section 6.4.3).

Here, $m \geq 1$ is a given constant denoting the number of components in the probability mixture. Such a model can account for a hidden multi-state risk structure and is therefore deemed appropriate for catastrophe-linked risks. See for instance Trottier et al. (2018) for a CAT bonds returns model providing evidence of the presence of multiple states or regimes for such financial products. The following notation will be used for the moment generating functions:

$$\zeta_i^{(t)} \equiv \mathbb{E}^{\mathbb{P}}\left[e^{tZ_i}\right], \qquad i \in \{1, \ldots, m\}, \quad t \in \mathbb{R}. \tag{6.9}$$

The Esscher transform is considered as it is a common choice in the literature (e.g., Gerber and Shiu, 1994). This transform is equivalent to the construction a pricing measure $\mathbb{Q}_X$ whose

Radon-Nikodym derivative is

$$\frac{d\mathbb{Q}_X}{d\mathbb{P}} \equiv \frac{X^a}{\mathbb{E}^{\mathbb{P}}[X^a]} = \frac{e^{a\sigma Z_I}}{\mathbb{E}^{\mathbb{P}}[e^{a\sigma Z_I}]}, \tag{6.10}$$

where $a \in \mathbb{R}$ is a constant and where we have $\mathbb{E}^{\mathbb{P}}[X^a] < \infty$. Here note that $\mathbb{Q}_X$ as well as the pricing kernel are risk-specific in this approach. The distribution of Z_I under $\mathbb{Q}_X$ is given in the following proposition proven in Appendix A.4.

Proposition 6.1. *We have*

$$\mathbb{Q}_X(Z_I \leq z) = \sum_{i=1}^{m} \tilde{p}_i \tilde{F}_i(z), \qquad z \in \mathbb{R}, \tag{6.11}$$

where, for all $i \in \{1, \ldots, m\}$,

$$\tilde{p}_i \equiv \frac{p_i \zeta_i^{(a\sigma)}}{\sum_{j=1}^m p_j \zeta_j^{(a\sigma)}}, \qquad \tilde{F}_i(z) \equiv \int_{-\infty}^{z} \tilde{f}_i(x)dx, \qquad \tilde{f}_i(x) \equiv \frac{e^{a\sigma x}}{\zeta_i^{(a\sigma)}} f_i(x). \tag{6.12}$$

The distortion operator g that applies the Esscher transform for the risk X, i.e., such that $g\big(\mathbb{P}(X > x)\big) = \mathbb{Q}_X(X > x)$ for all $x \in \mathbb{R}$, is given in the proposition below whose proof is a direct application of Proposition 3.2.

Proposition 6.2. *The distortion operator that performs the Esscher change of measure for* X *is*

$$g_{Z_I}^{\mathbb{Q}_X,\mathbb{P}}(u) \equiv \bar{F}_{Z_I,\mathbb{Q}_X}\big(\bar{F}^{-1}_{Z_I,\mathbb{P}}(u)\big), \qquad u \in [0,1]. \tag{6.13}$$

6.4.2 Recuperating distortions from previous literature

The above framework is general as it encompasses several distortion operators encountered in the literature. Some examples are given below for the case $m = 1$; detailed proofs are available from the authors upon request.

1. The PH transform (from Wang, 1995) $g_{Z_1}^{\mathbb{Q}_X,\mathbb{P}}(u) = u^{1-a\sigma}$ is obtained if Z_1 is a standard exponential variable under $\mathbb{P}$.

2. The Wang transform $g_{Z_1}^{\mathbb{Q}_X,\mathbb{P}}(u) = \Phi\big(\Phi^{-1}(u) + a\sigma\big)$ is obtained if $Z_1 \sim N(0,1)$ under $\mathbb{P}$.

3. The GB2 distortion $g_{Z_1}^{\mathbb{Q}_X,\mathbb{P}}(u) = 1 - \beta\big(\beta^{-1}(1-u;\tau,\alpha);\tau + a\sigma, \alpha - a\sigma\big)$ is obtained if e^{Z_1} follows a GB2$(\alpha, 1, 1, \tau)$ distribution (as defined in Klugman et al., 2012) under $\mathbb{P}$.

The multi-state extensions of the above distortions are also directly derived from the above general framework.

6.4.3 A normal-exponential mixture distortion

A simple approximation to the jump-diffusion distortion relying on a normal-exponential mixture is derived in this section. Consider the NCP distribution (4.12). Assume that $p = 1$, and that λ and η_1 are both small. η_1 being small ensures that the size of $\mathcal{Z}$ is negligible when compared to Y_1, whereas a small λ makes the probability of having more than a single jump low. Then, (4.12) can be approximated through

$$\begin{aligned}
\Pr\left(\mathcal{Z} + \sum_{i=1}^{P} Y_i \leq x\right) &\approx \Pr(P=0)\Phi\left(\frac{x-\mu}{\sigma}\right) + \underbrace{\Pr(P=1)}_{\approx 1-\Pr(P=0)} \Pr(\mathcal{Z} - Y_1 = 0) \\
&\qquad + \sum_{j=2}^{\infty} \underbrace{\Pr(P=j)}_{\approx 0} \Pr\left(\mathcal{Z} + \sum_{i=1}^{j} Y_i \leq x\right) \\
&\approx e^{-\lambda}\Phi\left(\frac{x-\mu}{\sigma}\right) + (1-e^{-\lambda})\Pr(\mathcal{Z} + Y_1 \leq x) \\
&\approx e^{-\lambda}\Phi\left(\frac{x-\mu}{\sigma}\right) + (1-e^{-\lambda})\Pr(Y_1 \leq x)\,,
\end{aligned}$$

which is consistent with a normal-exponential mixture. This justifies the use of such a mixture to construct a distortion operator approximating the behavior of the distortion (6.4).

The mixture we propose consists of $m = 2$ states such that $Z_1 \sim N(0,1)$ and $Z_2 \sim \text{Exp}(\eta)$ under $\mathbb{P}$,[14] where Exp(η) is the exponential distribution with inverse scale parameter $\eta > 0$.

[14] Note that we can consider $Z_1 \sim N(0,1)$ without loosing scale generality, since by (6.6) we are in fact considering a more general risk of the form $X = e^{\mu+\sigma Z_I} = e^{\mu+Z_I'}$, with $Z_1' \sim N(0,\sigma^2)$ and $Z_2' \sim \text{Exp}(\eta/\sigma)$.

It follows that

$$\mathbb{P}(Z_I \leq z) = p_1 \Phi(z) + (1 - p_1)(1 - e^{-\eta z}) \mathbb{1}_{\{z \geq 0\}}, \qquad z \in \mathbb{R}. \tag{6.14}$$

As shown in Appendix A.5, provided that $\eta > a\sigma$,

$$\begin{aligned} \mathbb{Q}_X(Z_I \leq z) &= \tilde{p}_1 \Phi(z - a\sigma) + (1 - \tilde{p}_1)\big(1 - e^{-(\eta - a\sigma)z}\big) \mathbb{1}_{\{z \geq 0\}}, \qquad z \in \mathbb{R}, \\ \tilde{p}_1 &= \frac{p_1 e^{\frac{1}{2}(a\sigma)^2}}{p_1 e^{\frac{1}{2}(a\sigma)^2} + (1 - p_1)\frac{\eta}{\eta - a\sigma}}. \end{aligned} \tag{6.15}$$

The corresponding distortion function is given by Proposition 6.1 and Proposition 6.2. It is characterized by the parameters (p_1, η, ξ) where $\xi \equiv a\sigma$, with domain $p_1 \in [0, 1]$, $\eta > 0$, $\xi < \eta$. The computation of this distortion is quite straightforward as it involves only basic functions. Moreover, it provides a more accurate representation of CAT bond spreads as illustrated in Figure 3. In particular, residuals in Figure 3(c) do not display systematic pricing errors in contrast with the aforementioned non-jump-diffusion based distortions. Hence, our mixed distribution approach is successful in providing an accurate description of observed CAT bond spreads while being simple to implement in practice.

7 Conclusion

We propose a general class of probability distortion operators which can be reconciled with multiple pricing principles used in finance and insurance such as no-arbitrage models, equilibrium models and actuarial premium calculation principles. Connections between distortion pricing and the latter pricing methods are illustrated. The Wang (2000) transform and its generalizations such as the NIG distortion of Godin et al. (2012) are recovered as particular cases in our study. The flexibility of the proposed approach allows deriving several new distortions based for instance on non-Gaussian financial theory which can integrate risk-aversion, distribution features (e.g., skewness and kurtosis) and other considerations that should be taken into account within pricing procedures. In particular, it is well-known that accounting for the heavy tail feature of the loss

distributions observed in non-life insurance is critically important for premium calculation and risk measurement.

An important area of research which is tackled through our work is the testing of catastrophe (CAT) bond pricing models from publicly available transaction information. Following the lead of Wang (2004) and Galeotti et al. (2013), we provide an empirical study which indicates that an exponential jump-diffusion distortion is adequate for explaining CAT bond spreads, but only if the distortion is allowed to incorporate risk-aversion to natural disasters. Also, a general yet simple class of probability distortion operators based on the Esscher change of measure for multi-state structured risks is proposed to price CAT risks. This new class of distortions provides an accurate depiction of observed CAT bond spreads while being straightforward to implement in practice. Such jump-diffusion and mixture-based distortion operators were constructed through the general approach presented in this paper.

Regarding extensions, it has been shown that the Wang transform can be used to obtain the Black-Scholes prices of exotic options (Labuschagne and Offwood, 2013). The Wang transform has also been extended to a multivariate setting (see, e.g., Kijima, 2006; Wang, 2007). It seems reasonable to presume that our framework can be extended in similar directions. Moreover, another potential application of our distortion operators is to produce, by way of equation (2.3), distortion-based risk measures that can be used for a wide range of applications, including capital allocation and optimal reinsurance (see Dowd and Blake, 2006, for an account of the applications of distortion risk measures).

A Proofs

A.1 Proof of Proposition 3.1

For $u \in \{0, 1\}$ the result is trivial because the essential supremum and infimum of Z are identical under both $\mathbb{P}$ and $\mathbb{Q}$ due to the equivalence of both measures. Since h is continuous and strictly

increasing, we have

$$\bar{F}^{-1}_{h(Z),\mathbb{P}}(u) = h\big(\bar{F}^{-1}_{Z,\mathbb{P}}(u)\big) \text{ for all } u \in (0,1),$$
$$\bar{F}_{h(Z),\mathbb{Q}}(z) = \bar{F}_{Z,\mathbb{Q}}\big(h^{-1}(z)\big) \text{ for all } z \in (\text{ess inf } h(Z), \text{ess sup } h(Z))$$

as shown by Property 1.5.16 of Denuit et al. (2006). Using these last two identities in Definition 3.2 gives

$$\forall\, u \in (0,1), \quad g^{\mathbb{Q},\mathbb{P}}_{h(Z)}(u) = \bar{F}_{h(Z),\mathbb{Q}}\big(\bar{F}^{-1}_{h(Z),\mathbb{P}}(u)\big) = \bar{F}_{Z,\mathbb{Q}}\big(\bar{F}^{-1}_{Z,\mathbb{P}}(u)\big) = g^{\mathbb{Q},\mathbb{P}}_{Z}(u).$$

A.2 Proof of Proposition 3.2

Define $a_h(x) \equiv \inf\{z : h(z) > x\}$. Then $h(z) > x$ if and only if $z > a_h(x)$.

$$\begin{aligned}
g^{\mathbb{Q},\mathbb{P}}_{Z}\big(\bar{F}_{X,\mathbb{P}}(x)\big) &= \bar{F}_{Z,\mathbb{Q}}\big(\bar{F}^{-1}_{Z,\mathbb{P}}\left(\bar{F}_{X,\mathbb{P}}(x)\right)\big) \\
&= \bar{F}_{Z,\mathbb{Q}}\big(\bar{F}^{-1}_{Z,\mathbb{P}}\left(\mathbb{P}\left(h(Z) > x\right)\right)\big) \\
&= \bar{F}_{Z,\mathbb{Q}}\big(\bar{F}^{-1}_{Z,\mathbb{P}}\left(\mathbb{P}\left(Z > a_h(x)\right)\right)\big) \\
&= \bar{F}_{Z,\mathbb{Q}}\big(\bar{F}^{-1}_{Z,\mathbb{P}}\left(\bar{F}_{Z,\mathbb{P}}\left(a_h(x)\right)\right)\big) \\
&= \bar{F}_{Z,\mathbb{Q}}\big(a_h(x)\big) \\
&= \bar{F}_{X,\mathbb{Q}}(x).
\end{aligned}$$

A.3 Proof of Theorem 3.1

The survival function of X distorted by $g^{\mathbb{Q},\mathbb{P}}_{Z}$ is $\bar{F}_{X,g^{\mathbb{Q},\mathbb{P}}_{Z}}(x) \equiv g^{\mathbb{Q},\mathbb{P}}_{Z}\big(\bar{F}_{X,\mathbb{P}}(x)\big)$. Using the chain rule to differentiate this equation with respect to x gives

$$f_{X,g^{\mathbb{Q},\mathbb{P}}_{Z}}(x) = \dot{g}^{\mathbb{Q},\mathbb{P}}_{Z}\big(\bar{F}_{X,\mathbb{P}}(x)\big) f_{X,\mathbb{P}}(x), \tag{A.1}$$

where $\dot{g}^{\mathbb{Q},\mathbb{P}}_{Z}(u) \equiv \frac{d}{du} g^{\mathbb{Q},\mathbb{P}}_{Z}(u)$.

By Definition 3.2, we have $g_Z^{\mathbb{Q},\mathbb{P}}(u) \equiv \bar{F}_{Z,\mathbb{Q}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(u)\big)$. Taking the derivative with respect to u yields

$$\dot{g}_Z^{\mathbb{Q},\mathbb{P}}(u) = -f_{Z,\mathbb{Q}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(u)\big)\frac{d\bar{F}_{Z,\mathbb{P}}^{-1}(u)}{du}. \tag{A.2}$$

Next, we note that

$$1 = \frac{d}{du}u = \frac{d}{du}\bar{F}_{Z,\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(u)\big) = -f_{Z,\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(u)\big)\frac{d\bar{F}_{Z,\mathbb{P}}^{-1}(u)}{du} \quad \Rightarrow \quad \frac{d\bar{F}_{Z,\mathbb{P}}^{-1}(u)}{du} = \frac{-1}{f_{Z,\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(u)\big)}.$$

Using this last equality in (A.2) yields

$$\dot{g}_Z^{\mathbb{Q},\mathbb{P}}(u) = \frac{f_{Z,\mathbb{Q}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(u)\big)}{f_{Z,\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(u)\big)} \equiv \xi_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(u)\big), \tag{A.3}$$

where we have used Definition 3.3. Using (A.3) in (A.1) concludes the proof.

A.4 Proof of Proposition 6.1

From (6.7),

$$\mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_I}\big] = \sum_{i=1}^{m}\mathbb{P}(I=i)\mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_I}\big|I=i\big] = \sum_{i=1}^{m}p_i\mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_i}\big] = \sum_{i=1}^{m}p_i\zeta_i^{(a\sigma)},$$

where definition (6.9) is used. Using this in (6.10) yields $\frac{d\mathbb{Q}_X}{d\mathbb{P}} = \frac{e^{a\sigma Z_I}}{\sum_{i=1}^m p_i\zeta_i^{(a\sigma)}}$. Hence,

$$\begin{aligned}
\mathbb{Q}_X(Z_I \le z) \equiv \mathbb{E}^{\mathbb{Q}_X}\big[\mathbb{1}_{\{Z_I\le z\}}\big] \equiv \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}_X}{d\mathbb{P}}\mathbb{1}_{\{Z_I\le z\}}\right] &= \frac{\mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_I}\mathbb{1}_{\{Z_I\le z\}}\big]}{\sum_{j=1}^m p_j\zeta_j^{(a\sigma)}},\\
&= \sum_{i=1}^{m}\frac{p_i\mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_i}\mathbb{1}_{\{Z_i\le z\}}\big]}{\sum_{j=1}^m p_j\zeta_j^{(a\sigma)}},\\
&= \sum_{i=1}^{m}\frac{p_i\zeta_i^{(a\sigma)}}{\sum_{j=1}^m p_j\zeta_j^{(a\sigma)}}\mathbb{E}^{\mathbb{P}}\left[\frac{e^{a\sigma Z_i}}{\zeta_i^{(a\sigma)}}\mathbb{1}_{\{Z_i\le z\}}\right],
\end{aligned}$$

from which one can indeed conclude (6.11).

A.5 Proof of Eq. (6.15)

Under $\mathbb{P}$, $Z_1 \sim N(0,1)$ and $Z_2 \sim \text{Exp}(\eta)$. Hence, the densities of Z_1 and Z_2 are respectively

$$f_1(x) = \phi(x) \equiv \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \qquad f_2(x) = \eta e^{-\eta x} \mathbb{1}_{\{x \geq 0\}}, \qquad x \in \mathbb{R}.$$

Furthermore,

$$\zeta_1^{(a\sigma)} \equiv \mathbb{E}^{\mathbb{P}}\left[e^{a\sigma Z_1}\right] = e^{\frac{1}{2}(a\sigma)^2}, \qquad \zeta_2^{(a\sigma)} \equiv \mathbb{E}^{\mathbb{P}}\left[e^{a\sigma Z_2}\right] = \frac{\eta}{\eta - a\sigma},$$

where the latter holds if $\eta > a\sigma$. It follows that

$$\tilde{f}_1(x) \equiv \frac{e^{a\sigma x}}{\zeta_1^{(a\sigma)}} f_1(x) = \phi(x - a\sigma), \qquad \tilde{f}_2(x) \equiv \frac{e^{a\sigma x}}{\zeta_2^{(a\sigma)}} f_2(x) = (\eta - a\sigma) e^{-(\eta - a\sigma)x} \mathbb{1}_{\{x \geq 0\}}, \qquad x \in \mathbb{R}.$$

Then, applying Proposition 6.1 yields (6.15).

B A simple example where $\mathbb{Q}$ and the distortion-based measure differ

This appendix provides a simple example where a martingale measure $\mathbb{Q}$ and the pricing measure $\tilde{\mathbb{Q}}$ defined as $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \equiv \xi_Z^{\mathbb{Q},\mathbb{P}}\left(\bar{F}_{Z,\mathbb{P}}^{-1}(\bar{F}_{X,\mathbb{P}}(X))\right)$ differ. Consider a market with a null risk-free rate $r = 0$ where the stock price is given by $S_t = S_0 \exp\left(\sum_{j=1}^{t} \epsilon_j\right)$, $t = 0, 1, 2$, where $\epsilon_j = \mu_j + \sigma_j z_j^{\mathbb{P}}$, and under $\mathbb{P}$, $z_1^{\mathbb{P}}$ and $z_2^{\mathbb{P}}$ are independent standard Gaussian random variables. Note that the convention $\sum_{j=1}^{0} \epsilon_j \equiv 0$ is used. Consider the martingale measure $\mathbb{Q}$ obtained through a discrete-time version of the Girsanov transform (see for instance Godin et al., 2017):

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\sum_{j=1}^{2} \lambda_j z_j^{\mathbb{P}} - \frac{\lambda_j^2}{2}\right), \quad \lambda_j \equiv -\frac{\mu_j + \frac{\sigma_j^2}{2}}{\sigma_j}.$$

Under $\mathbb{Q}$, ϵ_1 and ϵ_2 are independent with $\epsilon_j \sim N\left(-\frac{\sigma_j^2}{2}, \sigma_j^2\right)$.

Now consider $Z \equiv \epsilon_1 + \epsilon_2$ and $X \equiv S_2 = S_0 \exp(Z)$. Under $\mathbb{Q}$, $Z \sim N\left(-\sum_{j=1}^{2} \frac{\sigma_j^2}{2}, \sum_{j=1}^{2} \sigma_j^2\right)$

whereas $Z \sim N\left(\sum_{j=1}^{2}\mu_j, \sum_{j=1}^{2}\sigma_j^2\right)$ under $\mathbb{P}$. Therefore, from Remark 3.4 and (3.4),

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \equiv \xi_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{F}_{Z,\mathbb{P}}^{-1}(\bar{F}_{X,\mathbb{P}}(X))\big) = \xi_Z^{\mathbb{Q},\mathbb{P}}(Z) = \frac{f_{Z,\mathbb{Q}}(Z)}{f_{Z,\mathbb{P}}(Z)} = \exp\left(\zeta_1 Z - \frac{\zeta_2}{2}\right)$$

where $\zeta_1 \equiv -\frac{\mu_1+\mu_2+\frac{\sigma_1^2+\sigma_1^2}{2}}{\sigma_1^2+\sigma_2^2}$ and $\zeta_2 \equiv \frac{-(\mu_1+\mu_2)^2+\left(\frac{\sigma_1^2+\sigma_1^2}{2}\right)^2}{\sigma_1^2+\sigma_2^2}$. Assume $\mu_1 = \mu_2 = 0$ and $\sigma_1 \neq \sigma_2$. Then, clearly, $\tilde{\mathbb{Q}} \neq \mathbb{Q}$ since $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \neq \frac{d\mathbb{Q}}{d\mathbb{P}}$.

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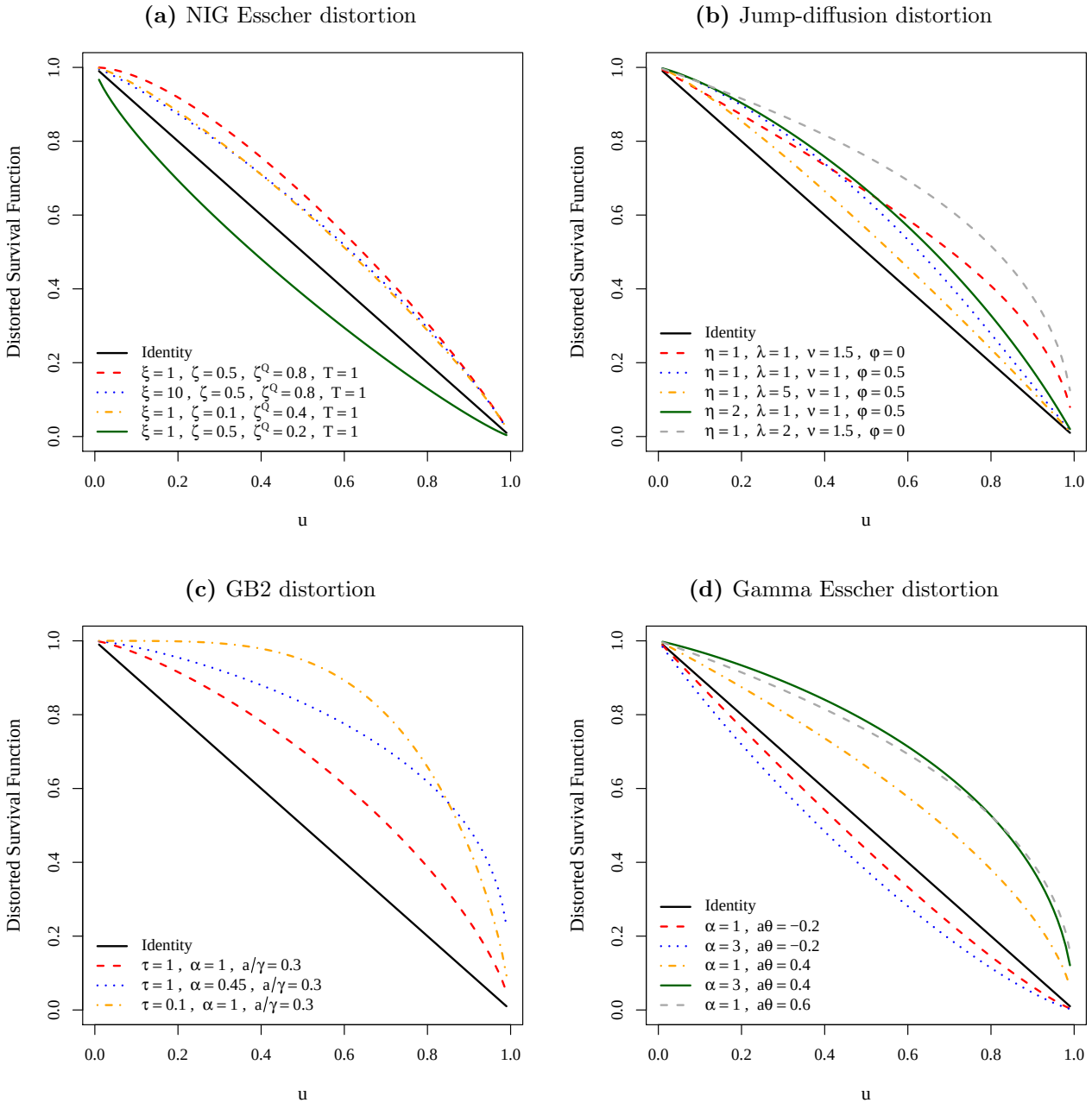
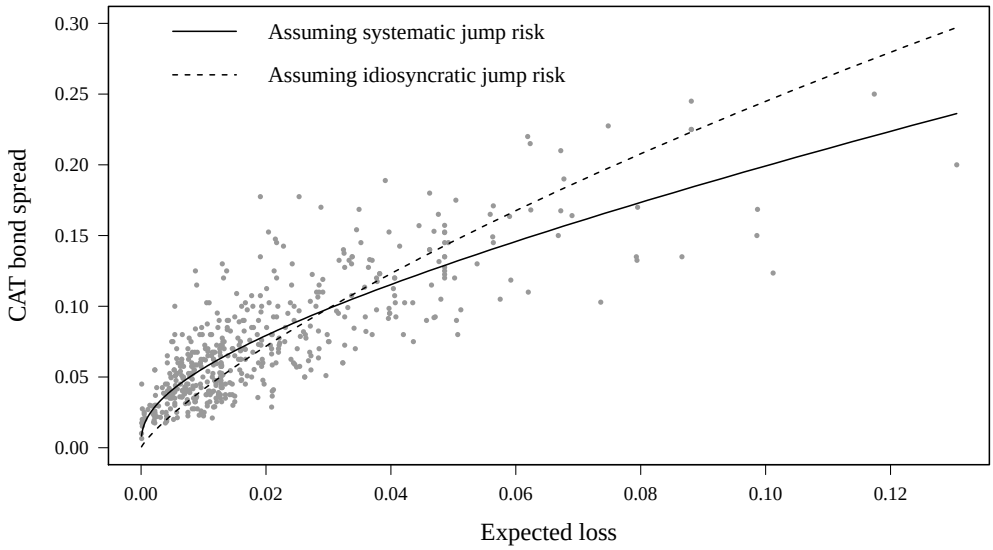
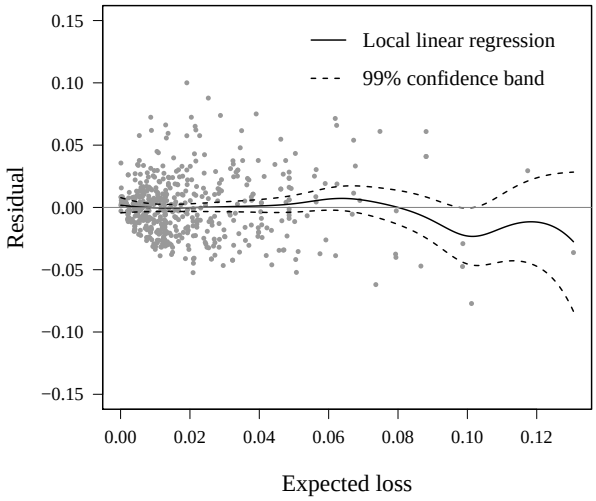


Figure 1: Visualization of the distorted survival function of a uniform $[0, 1]$ distribution, i.e., the function $u \to g_Z^{\mathbb{Q},\mathbb{P}}(1-u)$, for various distortions presented in the current paper with multiple sets of parameters. The distortions illustrated in the four sub-figures are **(a)** the NIG Esscher distortion from (4.5), **(b)** the jump-diffusion distortion from (6.4), **(c)** the GB2 distortion from (5.10), **(d)** the Gamma Esscher distortion from (5.13).

(a) Illustration of the calibrated distortion operators



(b) Residuals: Systematic jump risk ($\nu \geq 1$)



(c) Residuals: Idiosyncratic jump risk ($\nu = 1$)

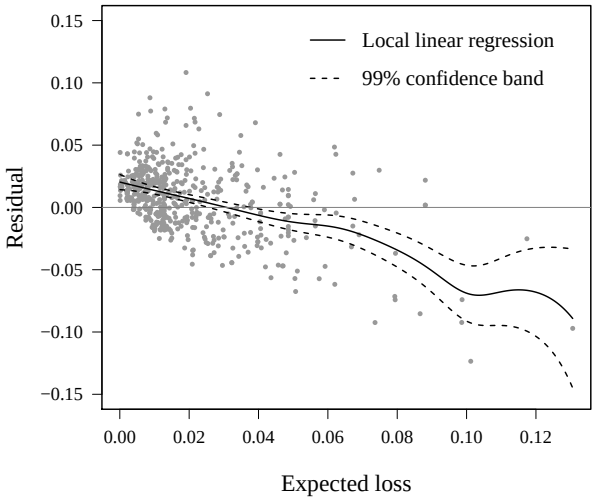
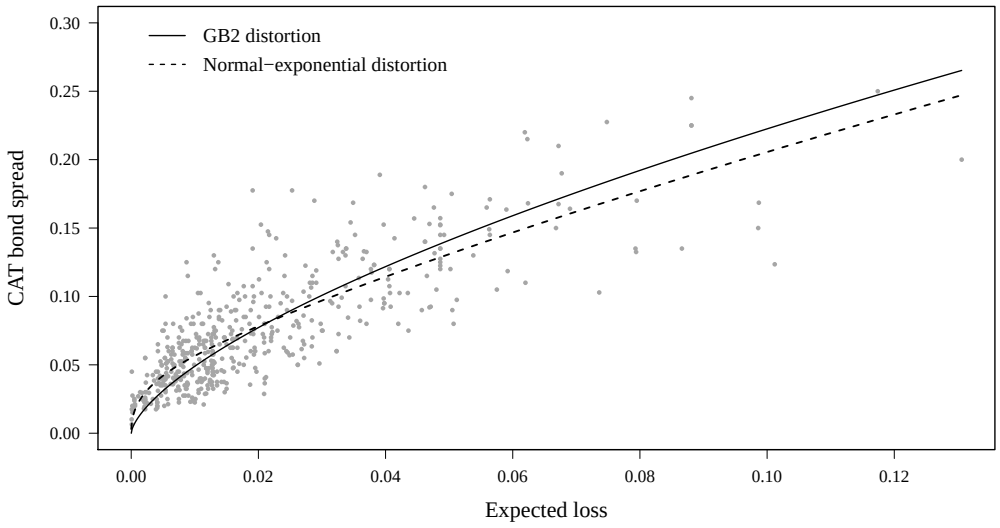
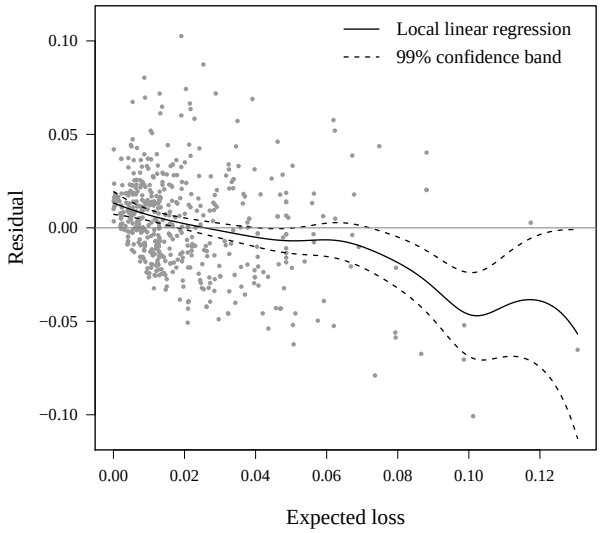


Figure 2: Calibration results for our Artemis.bm data set consisting of 508 CAT bond tranches issued between 1997 and 2016. The calibration is done by minimizing the sum of squared pricing errors. Graph **(a)** illustrates the empirical data and the calibrated distortion operator $g_{\varphi,\lambda,\eta,\nu}$ from (6.4). The dashed line is obtained under the constraint $\nu = 1$ (idiosyncratic jump risk), and the solid line is obtained without this constraint (i.e., with $\nu \geq 1$). The residuals are plotted against the expected loss in Graph **(b)** for the unconstrained case, and in Graph **(c)** for the constrained case. A local linear regression of the residuals against the expected loss is carried out to detect systematic pricing errors.

(a) Illustration of the calibrated distortion operators



(b) Residuals: GB2 distortion



(c) Residuals: Normal-exponential distortion

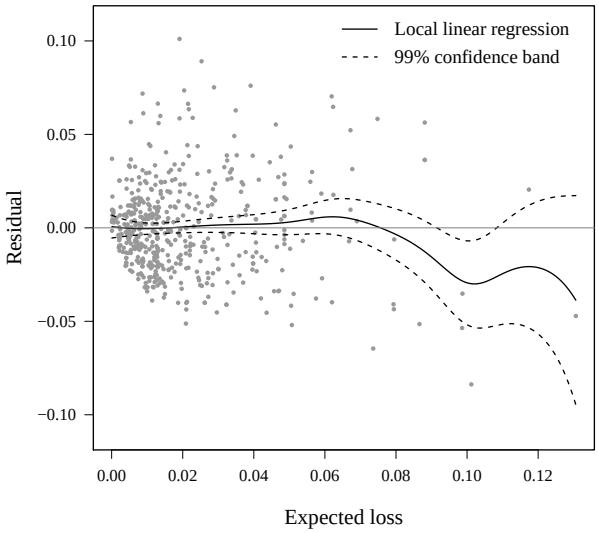


Figure 3: Calibration results for our Artemis.bm data set consisting of 508 CAT bond tranches issued between 1997 and 2016. The calibration is done by minimizing the sum of squared pricing errors. Graph **(a)** illustrates the empirical data and the calibrated distortion operators. The solid line corresponds to the GB2 distortion operator from (5.10), and the dashed line corresponds to the normal-exponential mixture distortion proposed in Section 6.4.3 of this paper. The residuals are plotted against the expected loss in Graph **(b)** for the GB2 distortion, and in Graph **(c)** for the normal-exponential distortion. A local linear regression of the residuals against the expected loss is carried out to detect systematic pricing errors.